A Relativistic Stochastic Process

Diploma Thesis

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Abstract

We give a review of the relativistic stochastic process denominated with the acronym ROUP, standing for relativistic Ornstein-Ohlenbeck process. This stochastic process was introduced in 1997 by Debbasch, Mallick and Rivet [*J. Stat. Phys. 88:945–966*] as a simplified model of irreversibility in a relativistic framework. This allows an investigation of the paradox arising when examining the large time and space regime of relativistic transport equations (like the relativistic Boltzmann equation) via the Chapman-Enskog approach, which inexplicably gives back non-relativistic parabolic equations.

We then prove the markovian irreversible character of this process even in an arbitrary curved space-time, by introducing a conditional entropy current based on the manifestly covariant Fokker-Planck formulation of the general relativistic ROUP.

Keywords : Relativistic stochastic processes, relativistic Brownian motion, Fokker-Planck equation

New, what do you own the world?
How do you own disorder?

S. Tankjan

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Introduction

The present diploma thesis deals with the construction of the so-called Relativistic Ornstein-Uhlenbeck process (denominated with the acronym ROUP), a stochastic process which was introduced in 1997 by Debbasch, Mallick and Rivet [*J. Stat. Phys. 88:945–966*] as a physically straightforward construction of a relativistic theory of Brownian motion for particles moving in a homogeneous, viscous medium. The scope of presenting such a physical-mathematical structure was to propose a simple model for irreversibility in a relativistic framework. It is a priori clear that such a model based on the Ornstein-Uhlenbeck stochastic model presents some evident limitations (like for example the fact that it does not take into account the formation of turbulence at high Reynolds numbers). However the ROUP is by construction very simple and straightforward, allowing a very deep and complete theoretical analysis, thus giving very useful insights on the behaviour of irreversibility in a relativistic scenario. The ROUP can indeed be formulated as a Fokker-Planck type transport equation for a 1-particle probability distribution function in a $4+4$-dimensional extended phase-space. This equation can be thought of as a simplified analogon of the relativistic Boltzmann equation [5], which is a controversial concepts but anyway reveals itself to be a valuable and widely used tool in astro-, plasma and nuclear physics. Still, it was not clear why the application of the so-called Chapman-Enskog approach [4] on this perfectly relativistic equation in the attempt to derive an approximated solution leads to thermomechanics theories which violate causality (like the covariant Eckart and Landau-Lifschitz relativistic theories of heat flow and viscosity). The idea proposed in [9] was to apply this same method on the much simpler equations describing the ROUP, and indeed the simpler structure of these equations allowed to formulate a satisfying explanation to that paradoxical situation (see Chapter V.1).

The second issue with regard to the ROUP which will largely be treated in this diploma thesis is an extension of this stochastic process to the framework of General Relativity. A manifestly covariant version of the Fokker-Planck type evolution equation of the ROUP will be proposed. Thank to this formulation it will be possible to introduce a conditional entropy 4-current and to prove an H-theorem for it in an arbitrary lorentzian curved space-time. This is a merely local result but it is enough to

show that irreversibility is maintained for the General Relativistic Ornstein-Uhlenbeck Process, for a very wide class of universes. This result can be related to the issue of the time's arrow (see reference [28]). It can be indeed interpreted as the statement that, in the physical context of the ROUP, any observer can single out a time direction (the future) in which the information content (represented by the conditional entropy) of a physical state (represented by a distribution function) degrades, tending to its minimum value.

Chapter I

Why a relativistic stochastic process?

I.1 Stochastic processes and galilean Brownian motion

The theory of stochastic processes plays a major role in many fields of modern physics, principally because of the elegance and flexibility with which it can cope with our ignorance with respect to the detailed description of various systems' dynamics, or our practical need to model and simplify them.

Brownian motion is probably the archetypical stochastic process, in this sense and also because of its popularity. This stochastic process was given the name of the English botanist R. Brown who, in 1827, had reported the observation of a very irregular motion displayed by a pollen particle immersed in a fluid. Exactly 100 years ago Einstein [15] and Smoluchowski [41] successfully treated the Brownian motion problem, also thanks the work David Bernoulli published in 1738. Through the works of Gibbs, Maxwell and Boltzmann [3, 20], statistical mechanics, as it grew out of the kinetic theory of gases, was the main area of application of probabilistic concepts in theoretical physics in the 19th century. Boltzmann in particular, putting forward the equation which now carries his name, was responsible of a very important contribution for statistical physics of non-equilibrium. His equation is a transport equation which describes the time evolution of the one-particle distribution function of a dilute fluid. Later it was realized that Boltzmann equation could be obtained by using the hypothesis of molecular chaos to truncate the so-called BBGKY[1] hierarchy, which relates the transport equations of the distribution functions for any number of particles

[1] Named after Born-Bogoliubov-Green-Kirkwood-Yvon

(see for instance reference [20]).

Applying a so-called Chapman-Enskog [4] expansion on Boltzmann equation it is possible to find the Navier-Stokes equations system, which describes the dissipative flow of newtonian fluids. This by the way means that this model should be realistic only near a (local or global) equilibrium state of the liquid.

The Brownian motion and all its variants are used – whether in physics, chemistry and biology or in finance [34], sociology and politics – to model a phenomenon (motion of the pollen particle, daily change in a stock market index) that is the outcome of many unpredictable and sometimes unobservable events (collisions with the particle of the surrounding liquid, buy/sell decisions of the single investor) which individually contribute a negligible amount to the observed phenomenon, but collectively lead to an observable effect. The details of the individual events may be impossible to consider, but their statistical properties (which in the end effectively determine the observed macroscopic behavior) may be known.

I.2 Special relativity

As it is well-known, in 1905 (his "Annus Mirabilis") Einstein put forward another milestone in modern physics with his work on Special Relativity [16]. Once this theory was fully accepted by the community it was natural to try to develop a relativistic version of the acquired classical theories, as it was done with electrodynamics (which was already compatible with Einstein's relativity) and mechanics. A full generalization of hydrodynamics was proposed only between 1940 and 1950 independently by Landau and Lifschitz, Eckart, and then by Lichnérowitz. These authors gave two relativistic versions of Euler and Navier-Stokes equations [26, 27, 21]. Both these generalizations belong to the so-called first order theories, which were conceived to model the dynamics of a relativistic ideal or dissipative fluid. They were called "first order" because their entropy currents contain no terms higher than first order in deviations from equilibrium (heat flow, viscous stresses, etc.).

Unfortunately it was soon realized that these first order theories reveal serious pathologies, violating Einstein principle of causality. This implies the even worse problem that these theories are unstable on a very short time scale, as it was proved by Hiscock and Lindblom [19], in the sense that they predict an evolution away from equilibrium in about the absurd short time-scales of $10^{-34}s$ for water at room temperature! It was then a necessity to replace these theories, and a natural way to do this was to go back to statistical physics.

The first works going in the direction of rebuilding statistical physics on the basis of Einstein's relativity are due to Jüttner [24], who generalized 1928 the celebrated Maxwell-Boltzmann distribution. Until now nobody succeeded in writing down a reasonable equivalent of the BBGKY hierarchy. It is in fact impossible, because of the finiteness of the speed of light signals, to construct a closed hierarchy of equations, having as unknown quantity functions of all particles' phase space coordinates at the same instant.

On account of these difficulties, in a statistical description of a macroscopic system involving many particles distribution functions, physicists concentrated their efforts on the notion of one-particle distribution function and on the attempt of constructing a transport equation verified by it.

The natural expression of the particle four-current in terms of the one-particle distribution function [5] strongly suggests that this quantity has to be a Lorentz scalar for the theory to be consistent within a relativistic framework. This fact is not at all trivial to show, and a critical reading of the existing literature on the subject offers a rather confusing perspective, often because of the wrong assumption that the phase-space volume is Lorentz-invariant. The proof of the Lorentz-covariance of the one-particle distribution function was given in a rigorous manner by Debbasch, Rivet and van Leeuwen [10] expressing this function as the expectation value of Dirac-delta distributions on the Lorentz-invariant statistical ensemble given by the concept of "micro-history".

Once the relativistic Boltzmann equation describing the evolution one-particle distribution function was known, it was possible, similarly to the galilean case, to derive a relativistic version of the Navier-Stokes system via a Chapmann-Enskog expansion. Noticeably the already mentioned first-order theories were found [21]. As already explained this theories contradict Einstein's relativity principle, whereas the relativistic Boltzmann equation seems to be exempt of all kind of pathologies. In other words the standard method which allows, in galilean physics, to obtain hydrodynamics equations starting from a statistical model produces aberrations if used in a relativistic framework.

The conclusion is that today there is no satisfying dynamic macroscopic theory of relativistic dissipative continuous media. Not only such a theory would be very useful, given the numerous situations in astrophysics and cosmology where such media show up, but it also seems important to try to understand the reasons of the impossibility to construct this kind of theory.

A plausible way to gain some insight into this problem is to momentarily abandon realistic physical models and examine toy-models, which, because of their simplicity, allow a deeper theoretical analysis.

I.3 Relativistic stochastic processes

Formally and conceptually galilean Brownian motion is probably the most simple irreversible known phenomenon. The quantity which undergoes an irreversible evolution is simply the particle density in physical space, which fulfills the canonical diffusion equation. This kind of evolution, equating second order spatial derivation (given by a laplacian) and first order time derivation, can be found also in other irreversible processes, like the Navier-Stokes equations system, where it describes momentum and energy diffusion. We should by the way note, that it is precisely because of the ubiquitous presence of this mathematical structure in the description of irreversible galilean phenomena that it is difficult to actualize a relativistic generalization. It is indeed

clear that space and time are treated asymmetrically by the diffusion equation.

So, on one hand Brownian motion is the most simple irreversible phenomenon we know, and on the other hand it also seems to own the core of any model of dissipative phenomenon in galilean fluid dynamics.

In this point of view, a relativistic generalization of galilean Brownian motion appears to be the most simple example of a relativistic irreversible phenomenon. It also provides a model for the construction of a coherent relativistic hydrodynamic theory, and an instrument to get some insight into the limits of the theories proposed in the past.

In 1997, Debbasch, Mallick and Rivet [8] proposed such a relativistic counterpart of the Brownian motion in the form of a relativistic variant of the Ornstein-Uhlenbeck process, the ROUP. We want to proceed to an extensive discussion of this stochastic process, but before it should be the case to rapidly illustrate some basics on stochastic processes in general, and on the galilean Brownian motion in particular (we invite to the consultation of reference [18] for an extensive introduction in probability theory).

Chapter II

Stochastic processes and stochastic differential equations

II.1 Reversible dynamical systems

Let us consider a dynamical system, whose time evolution is governed by the set of ordinary differential equation:

$$\frac{dx_i}{dt} = F_i(x), \qquad i = 1, \ldots, d \tag{II.1}$$

operating in a region of the **phase space** $X = \mathbb{R}^d$ with initial conditions $x_i(0) = x_i^0$. As we know, the evolution of such a system is fully deterministic, that is, the knowledge of the initial conditions x^0 at time $t = 0$ allows us to know the position of the system at any other time [37]. This kind of evolution is therefore said to be **reversible** or **invertible**, simply because the trajectory of the point $x(t)$ can be described, starting from the initial conditions x^0, by a non-selfintersecting (or intersecting but periodic) **dynamical law** $S_t : X \to X$, that is: $S_t(x^0) = x(t)$. The fact that the trajectory is nonintersecting with itself, allows us to reverse the dynamics completely unambiguously, i.e. $x^0 = S_{-t}(x(t))$.

In this case $x(\cdot)$ is of course a function of time giving us the position of the system in phase space $X = \mathbb{R}^d$.

Let us now introduce the concept of **distribution function** (or density) in a phase space X, which is an $L^1(X)$ function f with $f(x) \geq 1$ and $\|f\| \equiv \int_X |f|\, dx = 1$.

The distribution function f is assumed to describe the probability for the system

to be in a given phase space region $A \subset X$:

$$Prob(x \in A) = \int_A f(x)dx. \tag{II.2}$$

For the case of the deterministic system with known initial conditions described by equations (II.1) the distribution function at time t is trivially given by $f(t,x) = \delta(x - x(t))$. The concept of distribution function is much more useful when we have to do with stochastic nondeterministic evolution equations, in which case $x(t)$ is a random variable.

The evolution of the distribution function $f(t,x)$ is generally described by a so-called **Markov operator** $P : L^1 \to L^1$, that is $f(t,x) \equiv P^t f(0,x)$. A linear operator $P : L^1 \to L^1$ is called a Markov operator if it satisfies

1. $P^t f \geq 0$ and
2. $\left\|P^t f\right\| = \|f\|$

for all $t \in \mathbb{R}$ and $f \geq 0$.

It can be shown [28] that starting from an initial density $f(0,x)$, the evolution of the time dependent density $f(t,x) \equiv P^t f(0,x)$ is described by the **generalized Liouville equation**

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial (f F_i)}{\partial x_i}. \tag{II.3}$$

We remark that if the system of ordinary differential equations (II.1) is a Hamiltonian system,

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q_i}, \qquad i = 1, \ldots, s \end{cases} \tag{II.4}$$

where $2s = d$, and q and p are the position and momentum variables, $\mathcal{H}(p,q)$ is the system Hamiltonian, then equation (II.3) becomes

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\sum_{i=1}^{d} \frac{\partial (f F_i)}{\partial x_i} = -\sum_{i=1}^{s} \frac{\partial}{\partial q_i}\left(f \frac{\partial \mathcal{H}}{\partial p_i}\right) - \sum_{i=1}^{s} \frac{\partial}{\partial p_i}\left(-f \frac{\partial \mathcal{H}}{\partial q_i}\right) \\ &= -\sum_{i=1}^{s} \left[\frac{\partial f}{q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right], \end{aligned} \tag{II.5}$$

which is usually known as **Liouville equation** and written as:

$$\frac{df}{dt} = 0. \tag{II.6}$$

II.2 Irreversible dynamical systems

We saw in the previous section that a deterministic dynamical system can be inverted. This means that a necessary condition for our system to be irreversible, is that its dynamics is not described by deterministic differential equations, but we have to introduce a stochastic term.

Let us examine the behavior of the stochastically perturbed analog of equations (II.1), which we want to obtain adding a perturbation ξ_j, that should for example represent a random force by the environment acting on a particle. The following stochastic differential equation is often referred to as a nonlinear Langevin equation:

$$\frac{dx_i}{dt} = F_i(x) + \sum_{j=1}^{d} \sigma_{ij}(x)\xi_j(t), \qquad i = 1, \ldots, d \tag{II.7}$$

with the same initial conditions as before, where $\sigma_{ij}(x)$ is the amplitude of the stochastic perturbation and $\xi_j = \frac{dw_j}{dt}$ is a "white noise" term that is the formal derivative of a so-called **Wiener process**. "Formal derivative" because the Wiener process is not differentiable, as can be seen by its construction later in this section.

Stochastic processes

Examining equation (II.7), we observe that since $\xi(t)$ is a random variable for which we in principle know only statistical properties (see the definition of the Wiener process later) also $x(t)$ will be a random variable, whose statistics depends on that of $\xi(t)$. The quantities $\xi(\cdot)$ and $x(\cdot)$, which can be seen as a succession of indexed random variables, are referred to as **stochastic processes**.

Actually equation (II.7) has a formal status, not being mathematically well-defined because of two reasons. First of all the mathematical meaning of equation (II.7) is that of an equality between measures (see for example reference [31]) and should therefore preferably be cast in the following way:

$$dx_i = F_i(x)dt + \sum_{j=1}^{d} \sigma_{ij}(x)dw_j(t), \qquad i = 1, \ldots, d. \tag{II.8}$$

The second mathematical difficulty is given by the product in the term $\sigma_{ij}(x)dw_j(t)$. Again examining equation (II.7) we can imagine $\xi(t)$ being a random succession of pulses acting on the system giving rise to a pulse in $\frac{dx}{dt}$ and hence a jump in x. That has the effect that the value of x to be used in $\sigma_{ij}(x)$ is undetermined. The Itô convention [22, 23] assigns a meaning to (II.7) by adding, as a matter of definition, the rule that in $\sigma_{ij}(x)$ the value of x just before the pulse should be taken. Other authors assumed other convention, obtaining different but equivalent results. The most famous alternative to the Itô calculus is the one developed by Stratonovich, who proposed to take the value of x at the end of the pulse for $\sigma_{ij}(x)$. As a matter of fact that we just discussed is commonly known as Itô-Stratonovich dilemma (see for

an interesting pedagogical discussion reference [38]). What should be reassuring is that this dilemma is physically irrelevant, because it automatically disappears once we endow our stochastic differential equation with a microscopic picture of the noise represented by $w_j(t)$ telling us how to interpret it. Furthermore the stochastic noise is never perfectly white (that is not really a "succession of Dirac delta functions"), meaning that the Itô-Stratonovich dilemma doesn't even actually show up in physics.

The Wiener process or Brownian motion

Let us now properly define the **Wiener process**, whose formal derivative is the white noise $\xi(t) = \frac{dw}{dt}$ we used in equation (II.7). The Wiener process is also commonly known as **Brownian motion** when we take the phase space to be physical space. We will define the Wiener process giving its statical properties via its distribution function. This distribution function can be seen to satisfy the conditions of **Kolmogorov Extension Theorem** [31], which therefore guarantees the existence of such a stochastic process.

We say that a continuous process $\{w(t)\}_{t>0}$ is a one-dimensional **Wiener process** if the following two conditions are satisfied:

1. $w(0) = 0$ and
2. for all values of s and t, $0 \leq s \leq t$ the random variable $w(t) - w(s)$ has the gaussian distribution

$$g(t-s, x) = \frac{1}{\sqrt{2\pi(t-s)}}\, exp\left(-\frac{x^2}{2(t-s)}\right). \tag{II.9}$$

This definition is naturally extended in d-dimensions by creating a d-dimensional vector $w(t) = (w_1(t), \ldots, w_d(t))$ with joint density

$$g(t, x_1, \ldots, x_d) = g(t, x_1) \cdot \ldots \cdot g(t, x_d), \tag{II.10}$$

because of the independence of the increments. We can thus easily compute the first moments of the d-dimensional Wiener process:

$$\int_{\mathbb{R}^d} g(t,x)dx = 1, \tag{II.11}$$

$$\int_{\mathbb{R}^d} x_i g(t,x)dx = 0, \qquad i = 1, \ldots, d, \tag{II.12}$$

$$\int_{\mathbb{R}^d} x_i x_j g(t,x)dx = \delta_{ij} t, \qquad i,j = 1, \ldots, d. \tag{II.13}$$

(For an obliged and very extensive reference on Brownian motion we can refer to [35].)

Now that all elements of equation (II.7) are properly mathematically defined it is more than natural to try to find a solution of it. As said before, this solution

will be a stochastic process. Mathematicians have shown [17] that, as in the case of a nonperturbed system of ordinary differential equations, if the functions $F_i(x)$ and $\sigma_{ij}(x)$ are Lipschitz-continuous, then equation (II.7) has a unique solution.

Let us approximate solutions to equation (II.7) with a linear Euler extrapolation formula [28]. Suppose that the solution $x(t)$ is given on some interval $[0, t_0]$. Then for small values of Δt and using the Itô scheme, we may approximate x at time $t_0 + \Delta t$ using x at time t_0 with

$$x(t_0 + \Delta t) \simeq x(t_0) + F(x(t_0))\Delta t + \sigma(x(t_0))\Delta w, \tag{II.14}$$

with $\Delta w = w(t_0 + \Delta t) - w(t_0)$. This formula is knwon as the **Euler-Bernstein equation** because of the use of the Euler approximation by Bernstein in his original work on stochastic differential equations.

The Fokker-Planck equation

We now take a look to the statistics of the solution $x(t)$ to equation (II.7), which is described by the distribution function $f(t, y) \equiv Prob(x(t) = y)$.

To make sure that $f(t, x)$ exists and is differentiable we have to impose some conditions on the factors $\sigma_{ij}(x)$. Let us define the quadratic symmetric non-negative matrix

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x). \tag{II.15}$$

We now sketch the derivation of the evolution equation for $f(t, x)$ which technically requires that σ_{ij} and a_i are C^2 and that they and their first derivatives are bounded. the main idea is to calculate in two different ways the expectation value of a quantity and equate this two results to obtain what we want.

Assume $x(t)$ is the solution to equation (II.7) in $[0, t_0]$ for a $t_0 > 0$. Pick $\epsilon > 0$ and extend $x(t)$ on the interval $[t_0, t_0 + \epsilon]$ thanks to the Euler-Bernstein formula (II.14) by

$$x(t_0 + \Delta t) = x(t_0) + F(x(t_0))\Delta t + \sigma(x(t_0))\Delta w(t_0), \tag{II.16}$$

where $0 \leq \Delta t \leq \epsilon$ and $\Delta w(t_0) = w(t_0 + \Delta t) - w(t_0)$.

Consider a test function $h \in C_0^3(\mathbb{R}^d)$, with compact support. The quantity of which we are going to calculate the expectation value is $h(x(t_0 + \Delta t))$. To do this, we make the assumption that $x(t)$ has a distribution $f(t, x)$ for $t \in [0, t_0 + \Delta t]$. Then $x(t_0 + \Delta t)$ has a distribution $f(t_0 + \Delta t, x)$ and the expected value of $h(x(t_0 + \Delta t))$ is

$$E\Big(h\big(x(t_0 + \Delta t)\big)\Big) = \int_{\mathbb{R}^d} h(x) f(t_0 + \Delta t, x) dx. \tag{II.17}$$

On the other hand the Euler-Bernstein equation (II.16) allows us to write

$$h\big(x(t_0 + \Delta t)\big) = h\big(Q(x(t_0), \Delta w(t_0))\big), \tag{II.18}$$

where

$$Q(x,y) = x + F(x)\Delta t + \sigma(x)y. \tag{II.19}$$

Since the two random variables $x(t_0)$ and $\Delta w(t_0)$ are independent for all $\Delta t \in [0,\epsilon]$, the random pair $(x(t_0), \Delta w(t_0))$ has the distribution

$$f(t_0,x)g(\Delta t, y), \tag{II.20}$$

where g is the distribution (II.10) of a d-dimensional Wiener process. Thus we may once more calculate the expected value of $h(x(t_0+\Delta t))$ from equation (II.18) to yield

$$E\Big(h\big(x(t_0+\Delta t)\big)\Big) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x + F(x)\Delta t + \sigma(x)y) f(t_0,x)g(\Delta t, y)\ dx\ dy. \tag{II.21}$$

We now can equate equation (II.17) and (II.21)

$$\int_{\mathbb{R}^d} h(x)f(t_0+\Delta t, x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x + F(x)\Delta t + \sigma(x)y) f(t_0,x)g(\Delta t, y)\ dx\ dy, \tag{II.22}$$

and proceed to a Taylor expansion of h, to then divide throughout by Δt and take the limits as $\Delta t \to 0$. We however have to Taylor expand h up to second order, because equation (II.13) roughly tells us that somehow $\Delta w \approx (\Delta t)^{1/2}$, so that the quadratic term of the series will still have a linear, thus not negligible, contribution in Δt. Taylor expanding the right-hand side of equation (II.22) we therefore get:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x + F(x)\Delta t + \sigma(x)y) f(t_0,x)g(\Delta t, y)\ dx\ dy \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Bigg\{ h(x) + \sum_{i=1}^d \frac{\partial h(x)}{\partial x_i}\Big(F_i(x)\Delta t + \sum_{k=1}^d \sigma_{ik}(x)y_k\Big) \\
&\qquad + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 h(x)}{\partial x_i \partial x_j}\Big(F_i(x)\Delta t + \sum_{k=1}^d \sigma_{ik}(x)y_k\Big)\Big(F_j(x)\Delta t + \sum_{k=1}^d \sigma_{jk}(x)y_k\Big) \\
&\qquad + R_T(x)(\Delta t)^{3/2}\Bigg\} g(\Delta t, y) f(t_0,x)\ dx\ dy,
\end{aligned} \tag{II.23}$$

where $R_T(x)(\Delta t)^{3/2}$ is the remainder of the Taylor expansion:

$$R_T(x)(\Delta t)^{3/2} = \frac{1}{3!}\sum_{i,j,k=1}^d \left.\frac{\partial^3 h}{\partial x_i \partial x_j \partial x_j}\right|_{x+\epsilon A} A_i A_j A_k, \quad \epsilon \in]0,1[, \tag{II.24}$$

with $A = F(x)\Delta t + \sigma(x)y$. This remainder can actually be shown to be of order $(\Delta t)^{3/2}$ (see the appendix of reference [2]).

We now proceed to the integration with respect to y in equation (II.23), making use of equations (II.11) through (II.13) and obtain (plugging the result into equation (II.22)):

$$\begin{aligned}&\int_{\mathbb{R}^d} h(x) f(t_0+\Delta t, x) dx \\ &= \int_{\mathbb{R}^d} \left\{ h(x) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 h}{\partial x_i \partial x_j}\Big(F_i(x)F_j(x)(\Delta t)^2 + a_{ij}(x)\Delta t\Big) \right. \\ &\quad \left. + \sum_{i=1}^{d} \frac{\partial h}{\partial x_i} F_i(x)\Delta t + R_T(x)(\Delta t)^{3/2} \right\} f(t_0, x) dx,\end{aligned} \tag{II.25}$$

where the definition of $a_{ij}(x)$ equation (II.15) was used.

Taylor expanding also the left-hand side of equation (II.25), dividing throughout by Δt and taking the limit $\Delta t \to 0$ we get:

$$\int_{\mathbb{R}^d} h(x) \frac{\partial f}{\partial t} dx = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 h}{\partial x_i \partial x_j} a_{ij}(x) + \sum_{i=1}^{d} \frac{\partial h}{\partial x_i} F_i(x) \right\} f(t_0, x) dx. \tag{II.26}$$

Integrating by parts the right-hand side of the last equation under the assumption that h has compact support we can rewrite the result as

$$\int_{\mathbb{R}^d} h(x) \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{d} \frac{\partial [F_i(x) f]}{\partial x_i} - \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x) f]}{\partial x_i \partial x_j} \right\} dx = 0, \tag{II.27}$$

and we thus finally obtain the evolution equation for the distribution $f(t, x)$ setting the term within braces identically zero:

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial [F_i(x) f]}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x) f]}{\partial x_i \partial x_j}. \tag{II.28}$$

This evolution equation is known as the (forward) **Fokker-Planck equation** or the **forward Kolmogorov equation** and plays a major role in investigations on the effects of random perturbations on the evolution of distribution functions, and in nonlinear phenomena in general. The first term on the right-hand side is usually called a drift term, while the second is known as a diffusion term.

The Fokker-Planck equation (II.28) is sometimes written in the equivalent form

$$\frac{\partial f}{\partial t} = \mathcal{L}_{FP} f. \tag{II.29}$$

where the Fokker-Planck differential operator is defined by

$$\mathcal{L}_{FP} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} F_i(x) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(x). \tag{II.30}$$

The **backward Fokker-Planck** equation would then be written as

$$\frac{\partial f}{\partial t} = \mathcal{L}_{FP}^{\dagger} f, \tag{II.31}$$

where the operator $\mathcal{L}_{FP}^{\dagger}$ is given by

$$\mathcal{L}_{FP}^{\dagger} = \sum_{i=1}^{d} F_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{II.32}$$

and is the adjoint operator to $\mathcal{L}_{FP}$ on the space of the square-integrable and twice continuous differentiable functions.

Another instructive way to derive Fokker-Planck equation is via the **Kramers-Moyal expansion** of the master equation by truncating it after second order [34, 39]. This expansion is practically a way to rewrite the integro-differential master equation into a partial differential equation of infinite order.

II.3 Conditional entropy

A useful concept to characterize the irreversibility of a process is given by the concept of **entropy**, in the sense of a quantity which is never decreasing in time, and thus somehow singles out the future from the past. Since a reversible system shows a kind of evolution which does not strictly do this distinction between past and future, we see that we want the entropy to be constant if the system evolves under an invertible Markov operator. These considerations lead us to the conclusion that the entropy functional represented by the **Boltzmann-Gibbs entropy** defined by

$$H_{BG}(f) = -\int_X f \ln f dx, \tag{II.33}$$

is not the functional we are seeking for. As a matter of fact, it can be shown [28], that the Boltzmann-Gibbs entropy can vary under a reversible evolution, and even decrease in time.

We address our attention to a generalization of the Boltzmann-Gibbs entropy, which doesn't present this kind of defects, and introduce the concept of conditional entropy.

If f and g are two distribution functions such that the support of f is in the support of g, $supp(f) \subset supp(g)$, then the **conditional entropy** of f with respect to g is defined by

$$H_c(f|g) = -\int_X f(x) \ln \frac{f(x)}{g(x)} dx. \tag{II.34}$$

Some of the properties of $H_c(f|g)$ are that:

1. Since f and g are distribution functions, $H_c(f|g) \leq 0$, because

$$H_c(f|g) = -\int_X f(x)\big(\ln f(x) - \ln g(x)\big)dx \leq -\int_X f(x)\big(\ln g(x) - \ln g(x)\big)dx = 0, \tag{II.35}$$

where the **integrated Gibbs inequality** was used:

$$-\int_X f(x)\ln f(x)dx \leq -\int_X f(x)\ln g(x)dx. \tag{II.36}$$

Equality in (II.35) holds if and only if $f \equiv g$.

2. If g is the constant density of the microcanonical ensemble, i.e., $g = 1/\mu_L(X)$, where $\mu_L(X)$ is the Lebesgue-measure of X, then $H_c(f|g) = H_{BG}(f) - \ln \mu_L(X)$. If the space X is normalized, then $g \equiv 1$ and $H_c(f|1) = H_{BG}(f)$. This explains in which sense the conditional entropy is a generalization of the Boltzmann-Gibbs entropy.

3. Let P be a Markov operator. Then

$$H_c(P^t f|P^t g) \geq H_c(f|g) \tag{II.37}$$

for $f \geq 0$, and all distribution functions g. This theorem was remarkably demonstrated by Voigt in reference [40].

4. From the definitions (II.34) and (II.33) it follows that

$$H_c(f|g) = H_{GB}(f) - H_{GB}(g) + \int_X \big(f(x) - g(x)\big)\ln g(x)dx. \tag{II.38}$$

Replacing f by $P^t f$ and g with a stationary distribution f_* of P (i.e. $P^t f_* = f_*$), we have

$$H_c(P^t f|f_*) = H_{GB}(P^t f) - H_{GB}(f_*) + \int_X \big(P^t f(x) - f_*(x)\big)\ln f_*(x)dx. \tag{II.39}$$

If we now suppose that $\lim_{t\to\infty} P^t f = f_*$, and that the Boltzmann-Gibbs entropy $H_{BG}(f)$ is maximized by the distribution f_*, then we conclude that the conditional entropy will be zero whenever the Boltzmann-Gibbs entropy is at its maximum value of $H_{BG}(f_*)$. We are thus lead to think of the conditional entropy as the difference between the thermodynamic entropy and the maximal equilibrium entropy, that is $H_c \sim \Delta S$.

The stationary distribution function f_*, if it exists and is unique, is also the state of maximal entropy, and the state toward which the dynamical system will tend. This means that it represents thermodynamical equilibrium.

Chapter III

The galilean Ornstein-Uhlenbeck process

III.1 The Brownian motion

As it was said before, taking the phase space X of the dynamical system described by equation (II.7) to be the physical space $\mathbb{R}^3$, we obtain what we usually call the Brownian motion already cited in the previous chapter. Let us be more specific on this point.

Let us suppose, that we want to model the diffusion process of one "test" particle in a fluid in classical galilean physics. We assume that the fluid is in a state of thermodynamic equilibrium, so that we can give a temperature T_{eq} and an inertial coordinate system $\mathcal{R}_{eq}$ in which the fluid is globally at rest. We will initially study our diffusion in this coordinate system $\mathcal{R}_{eq}$.

We already gave hints to the fact that the Wiener process represents our mathematical model for the Brownian motion. Let us for instance consider the case where our (pollen) "test" particle is in the rest coordinate system of the fluid $\mathcal{R}_{eq}$. The random collisions of the (light) fluid molecules causes the position of our pollen particle to vary stochastically in time. We assume that every collision happens instantly and is uncorrelated to the others. With this assumptions a good mathematical description of our model is indeed given by the Wiener process defined previously. But with one minor difference which is given by the fact that we have to introduce a length scale a and a time scale τ, treating a physical phenomenon (before the Wiener process $w(\cdot)$ was implicitly supposed to be indexed by an dimensionless parameter). The trajectory $x(t)$ of our pollen particle starting at $x = 0$ in $t = 0$ is thus traditionally written

as:

$$x(t) = a \cdot \mathbf{w}(t/\tau), \tag{III.1}$$

where $\mathbf{w}(\cdot) = \big(w_1(\cdot), w_2(\cdot), w_3(\cdot)\big)$, each $w_i(\cdot)$ being a Wiener process.

Choosing $n(t,x)$ to denote the probability density to find the test particle in position x at instant t, and noting that equation (III.1) is a special case of (II.7) for which we know the Fokker-Planck equation (II.28), we obtain

$$\frac{\partial n}{\partial t} = \chi \Delta n, \qquad t > 0, \tag{III.2}$$

which is the usual diffusion equation, and where Δ is the laplacian and the coefficient χ is simply related to the other characteristic dimensions of the problem by the relation

$$\chi = \frac{a^2}{\tau}. \tag{III.3}$$

This model, which is mainly due to Einstein [15], presents the problem that the path of the Borwnian motion is not differentiable (see reference [34]). A consequence of this fact is that it is not possible to define a velocity by the usual relation $\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$. This means in particular that the model of galilean diffusion presented here does not allow one to consider the probability distribution of the velocity or of the kinetic energy of the diffusing particle. In this context these concepts do not even make sense.

III.2 The galilean Ornstein-Uhlenbeck process

That is the reason why, after the works of Langevin, Ornstein and Uhlenbeck introduced in 1930 the stochastic process which today carries their names. The main idea is to add the uncertainty given by the white noise to the velocity, rather than to the position, as it was the case with the Brownian motion.

Being more specific, the Ornstein-Uhlenbeck is defined by two differential equations which fix the time derivatives of the position and the velocity (or the momentum). These are:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \dfrac{1}{m}\ \mathbf{p}(t) \\ \dfrac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{p}) + \mathcal{D}\dfrac{d\mathbf{w}}{dt}. \end{cases} \tag{III.4}$$

We should perhaps precise that the adjective "**galilean**" simply underlines the fact that the framework in which the Ornstein-Uhlenbeck process described by equations (III.4) takes place is *galilean physics*, in contraposition to *relativistic physics*. The term galilean is surely *not* meant to signify that the system of equation (III.4) is invariant under the group of galilean transformations. It is in fact clear that the system of equations (III.4) will not be invariant under a galilean boost $\mathbf{p} \to \mathbf{p} + \mathbf{p}'$.

This simply because they describe the motion of a particle in a fluid the preferred reference frame where this fluid is at rest [1].

In the first equation, which is only the galilean definition of the momentum, m is the mass of the diffusing particle. The second equation of (III.4) fixes the total force acting on the particle, which is the sum of a mean value depending on the the momentum of the particle with respect to the surrounding fluid, and a stochastic deviation around this mean value which depends only on the time and is represented by the Wiener process. The coefficient $\mathcal{D}$ has the dimension of momentum and plays a similar role to that played by the coefficient a in the previous section.

In the Ornstein-Uhlenbeck model velocity and momentum are well-defined at any time. This solves the problem present in the model given by the Brownian motion, where these quantities could not be calculated.

We will see that the form of the mean force F is completely determined by the statistics of the diffusing particles at equilibrium. This is intimately related to the **fluctuation-dissipation theorem**. We can in fact understand that equilibrium is reached depending on a kind of balance between the force $\mathbf{F}$, which tends to dissipate the particles' energy, and the stochastic noise, characterized by $\mathcal{D}$ and τ, which represents the particles' energy gain given by the fluid. If for instance we choose the Maxwell-Boltzmann distribution as the equilibrium distribution, we are forced to take a linear dissipation force $\mathbf{F}(\mathbf{p}) = -\alpha\mathbf{p}$, with a constant $\alpha > 0$. In this case it is found that the system of stochastic differential equations (III.4) is exactly integrable. However, we are not interested in this result, but we rather want to examine the case where a great number of particles diffuse. If we assume that all these particles are identical and do not directly interact (which comes down to supposing that the diffusing particles are sufficiently dilute in the surrounding fluid), it seems reasonable to model their diffusion thanks to the Ornstein-Uhlenbeck process. Now, the most convenient way to concretely describe this diffusion is a distribution function $\Pi(t, \mathbf{x}, \mathbf{p})$, which gives the probability density to find a particle in a region of the one-particle phase space. The transport equation of this distribution function can be found with the methods we already discussed.

III.3 Transport equation for the galilean Ornstein-Uhlenbeck process

Starting from equation (III.4), and going through the same kind of calculations that lead from (II.7) to the Fokker-Planck equation (II.28), we find that the probability density $\Pi(t, \mathbf{x}, \mathbf{p})$ obeys the following type of Fokker-Planck equation:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\mathbf{p}}{m} \Pi \right) + \frac{\partial}{\partial \mathbf{p}} \cdot (\mathbf{F}(\mathbf{p})\Pi) = \frac{\mathcal{D}^2}{\tau} \frac{\partial^2 \Pi}{\partial \mathbf{p}^2}, \qquad t > 0, \tag{III.5}$$

where the coefficient $\mathcal{D}^2/\tau$ plays a similar role to that played by χ in the usual diffusion equation. This equation is sometimes referred to as the **forward Kolmogorov**

[1] I am grateful to Professor Fröhlich for pointing me out this possible misunderstanding

equation or also as the **Kramers equation**.

Let us give some qualitative insights on the significance of equation (III.5). As it could be inferred from the derivation the general Fokker-Planck equation (II.28) in Section II.2, the right-hand side of equation (III.5) is due to the noise, expressed in the equations of motion (III.4) by the time derivative of the Wiener process $\mathbf{w}$. In a deterministic system the right-hand side would disappear and we would simply have the following simple evolution equation:

$$\frac{\partial \Pi}{\partial t} = -\frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\mathbf{p}}{m} \, \Pi \right) - \frac{\partial}{\partial \mathbf{p}} \cdot (\mathbf{F}(\mathbf{p})\Pi), \qquad t > 0, \tag{III.6}$$

whose significance can be clarified by the following calculations.

Let us define the reversible dynamical law $S_t : X \to X$, with $X = \{z = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3\}$ being our phase space, similarly as in Section II.1:

$$\begin{cases} z(t) = S_t(z^0) \\ z(0) = S_0(z^0) = z^0. \end{cases} \tag{III.7}$$

which represents the equations of motion

$$\frac{dz_i}{dt} = \widetilde{F}_i(z), \qquad i = 1, \ldots, 6 \tag{III.8}$$

where in this special case $\widetilde{F}(z) = (\frac{\mathbf{p}}{m}, \ \mathbf{F}(\mathbf{p}))$.

At this point, given the probability distribution function $\Pi \in L^\infty$ for the particle at time $t = 0$, we can define a so-called **Koopman operator** $K^t : L^\infty \to L^\infty$ by

$$K^t \Pi(z^0) = \Pi(S_t(z^0)) = \Pi(z). \tag{III.9}$$

Deriving equation (III.9) with respect to t we get:

$$\begin{aligned} \frac{\partial}{\partial t}\Pi(z) &= \frac{\partial}{\partial t}\Pi(S_t(z^0)) = \sum_i \dot{z}_i \frac{\partial}{\partial z_i}\Pi(S_t(z^0)) \\ &= \sum_i \dot{z}_i \frac{\partial}{\partial z_i}\Pi(z) = \sum_i \widetilde{F}_i \frac{\partial}{\partial z_i}\Pi(z) \\ &= \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}}\Pi(\mathbf{x}, \mathbf{p}) + \mathbf{F}(\mathbf{p}) \cdot \frac{\partial}{\partial \mathbf{p}}\Pi(\mathbf{x}, \mathbf{p}), \end{aligned} \tag{III.10}$$

which is the evolution equation for $\Pi(z) = K^t \Pi(z^0)$. En passant we note that the right-hand side of equation (III.10) is the application of the adjoint Fokker-Planck operator $\mathcal{L}^\dagger_{FP}$ defined in equation (II.32). Equation (III.10) is thus a special case of *backward* Fokker-Planck equation. It is indeed an evolution equation backward in time.

An evolution *forward* in time is expressed by a so-called Froebenius-Perron operator which, like the Koopman operator, is a kind of Markov operator (see reference [28]). If S_t is a nonsingular transformation (that is, if $\mu(A)$ implies $\mu(S_t(A))$ for

any Set A, μ being the Lebesgue measure), then the **Froebenius-Perron operator** $P^t : L^1 \to L^1$ associated to S_t is defined by:

$$\int_A P^t \Pi(z) dz = \int_{S_t^{-1}(A)} \Pi(z) dz. \tag{III.11}$$

Denoting the characteristic function of the set $A \subset \mathbb{R}^d$ as $1_A(z)$ (that is, $1_A(z) = 1$ for $z \in A$, and $1_A(z) = 0$ for $z \notin A$) we note the following simple fact:

$$\begin{aligned} \int_{\mathbb{R}^d} P^t \Pi(z) 1_A(z) dz &= \int_A P^t \Pi(z) dz = \int_{S_t^{-1}(A)} \Pi(z) dz \\ &= \int_A \Pi(S_t(z)) dz = \int_A K^t \Pi(z) dz = \int_{\mathbb{R}^d} \Pi(z) K^t 1_A(z) dz, \end{aligned} \tag{III.12}$$

and, because any distribution function can be constructed as the limit of a series of characteristic functions, this means that the Forebenius-Perron operator P^t is adjoint to the Koopman operator K^t in the space L^∞:

$$\langle P^t \Pi_1, \Pi_2 \rangle \equiv \int_{\mathbb{R}^d} P^t \Pi_1(z) \Pi_2(z) dz = \int_{\mathbb{R}^d} \Pi_1(z) K^t \Pi_2 dz \equiv \langle \Pi_1, K^t \Pi_2 \rangle. \tag{III.13}$$

With these remarkable result we can go back to equation (III.10)

$$\frac{\partial}{\partial t}(K^t \Pi) = \sum_i \widetilde{F}_i \frac{\partial}{\partial z_i} \Pi, \tag{III.14}$$

and use it combination with equation (III.13):

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}(P^t \Pi_1), \Pi_2 \right\rangle &= \frac{\partial}{\partial t} \langle P^t \Pi_1, \Pi_2 \rangle = \left\langle \Pi_1, \frac{\partial}{\partial t}(K^t \Pi_2) \right\rangle \\ &= \left\langle \Pi_1, \sum_i \widetilde{F}_i \frac{\partial}{\partial z_i} \Pi_2 \right\rangle \\ &= \sum_i \int_{\mathbb{R}^d} \left[\frac{\partial(\Pi_1 \Pi_2 \widetilde{F}_i)}{\partial z_i} - \Pi_2 \frac{\partial(\Pi_1 \widetilde{F}_i)}{\partial z_i} \right] dz \\ &= - \left\langle \sum_i \frac{\partial}{\partial z_i}(\Pi_1 \widetilde{F}_i), \Pi_2 \right\rangle, \end{aligned} \tag{III.15}$$

where we used partial integration exploiting the fact that Π_2 has compact support. We thus have

$$\frac{\partial}{\partial t}(P^t \Pi) = - \sum_i \frac{\partial}{\partial z_i}(\Pi \widetilde{F}_i) = - \frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\mathbf{p}}{m} \Pi \right) - \frac{\partial}{\partial \mathbf{p}} \cdot (\mathbf{F}(\mathbf{p}) \Pi), \tag{III.16}$$

which is exactly equation (III.6).

Again, as it was explained at the end of Section II.2 for the general Fokker-Planck equation (II.28), equation (III.5) can be written as

$$\frac{\partial \Pi}{\partial t} = \mathcal{L}_{FP}\Pi, \tag{III.17}$$

where the differential Fokker-Planck operator $\mathcal{L}_{FP}$, which is said to be the generator of our stochastic process, is defined by

$$\mathcal{L}_{FP} = -\frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{F}(\mathbf{p}) + \frac{\mathcal{D}^2}{\tau}\frac{\partial^2}{\partial \mathbf{p}^2}. \tag{III.18}$$

Now, in equation (III.5) we still have to choose the mean force $\mathbf{F}$ to fully characterize the galilean Ornstein-Uhlenbeck process. As said before, fixing the equilibrium distribution comes down to choosing a particular form for $\mathbf{F}$ by means of the fluctuation-dissipation theorem. We expect that, waiting long enough for the particles to be in thermal equilibrium with the surrounding fluid, the distribution evolves to a spatially homogenous state with a defined temperature corresponding to the equilibrium temperature T_e of the fluid. We thus reasonably assume that the equilibrium distribution Π_{eq} is given by the **Maxwell-Boltzmann distribution**:

$$\Pi_{eq}(\mathbf{p}) = (2\pi m k_B T_e)^{-\frac{3}{2}} \exp\left(-\frac{\mathbf{p}^2}{2mk_BT_e}\right), \tag{III.19}$$

which therefore has to be solution of equation (III.5). This forces $\mathbf{F}$ to be linear in momentum:

$$\mathbf{F}(\mathbf{p}) = -\alpha\frac{\mathbf{p}}{m}, \tag{III.20}$$

where the friction coefficient α, is given by:

$$\alpha = \frac{\mathcal{D}^2}{mk_BT_e}\frac{1}{\tau}. \tag{III.21}$$

This relation is referred to as a fluctuation-dissipation theorem, because it relates the characteristic parameter α of the dissipation force $\mathbf{F}$, with the characterizing parameters of stochastic noise $\mathcal{D}$ and τ. Given the fact that only the combination $\mathcal{D}^2/\tau$ appears in our equations it is common use to introduce the notation $D = \mathcal{D}^2/\tau$.

At this point we should do some remarks on the particular form of the dissipative force $\mathbf{F}$, which is sometimes justified by the observation that it is equal to the Stokes force acting in Navier-Stokes hydrodynamics on a sphere in a newtonian fluid. This observation is not completely pertinent, because it neglects the fact that this result is an approximation for small Reynolds numbers. We refer to [6] (from which this diploma thesis has greatly profited in general) and to the article [8] for an account on this point and to the Oseen corrections to Stokes law. What we want to retain anyway is that what we are investigating is just a simplified toy-model of irreversibility, which does not have any pretension to be realistic. We only impose that our model is simple enough to allow a deep theoretical analysis, and in this point of view we are almost forced to choose a white noise as a stochastic term, being the most tractable. If we further choose the Maxwell-Boltzmann distribution as equilibrium distribution, then we are automatically lead to the expression for $\mathbf{F}$, as explained before.

Chapter IV

The relativistic Ornstein-Uhlenbeck process

IV.1 Construction of the ROUP

In reference [8] the authors introduced the first explicit relativistic stochastic process, the relativistic Ornstein-Uhlenbeck process (ROUP), generalizing the galilean Ornstein-Uhlenbeck process to the framework of special relativity. We already had a glance of the advantages of the Ornstein-Uhlenbeck process over Brownian motion, which presents a parabolic structure that is incompatible with the fact that any speed should be limited by c, the vacuum speed of light (see [29]). In this chapter we would like to briefly sketch the construction of the ROUP, always referring to the original paper [8] for any detail.

We start by modifying equations (III.4) with (III.20) to make them compatible with special relativity:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \dfrac{1}{m}\,\dfrac{\mathbf{p}}{\gamma(\mathbf{p})} \\ \dfrac{d\mathbf{p}}{dt} = -\alpha(\mathbf{p})\dfrac{\mathbf{p}}{\gamma(\mathbf{p})} + \sqrt{2D}\dfrac{d\mathbf{w}}{dt}, \end{cases} \tag{IV.1}$$

where the Lorentz factor $\gamma(\mathbf{p})$ is given by the usual expression

$$\gamma(\mathbf{p}) = \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}, \tag{IV.2}$$

and $\mathbf{w} = (w_1(t), w_2(t), w_3(t))$ designates the three-dimensional Wiener process, as it was presented in section II.2. The first equation of (IV.1) is just the relativistic

relation between momentum and velocity. The second equation deserves some more discussion. It was derived imposing that in the global rest coordinate frame of the surrounding fluid be a gaussian white noise, like in the galilean case.

Starting from equations (IV.1) and going through a similar process to the one that lead from the stochastic equations of motion (II.7) to the Fokker-Planck equation (II.28), we obtain the following relativistic forward Kolmogorov equation (or relativistic Kramers equation) equation for the distribution function $\Pi(t, \mathbf{x}, \mathbf{p})$ in phase-space (again we refer to the original paper [8] for the details in the derivation):

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\mathbf{p}}{\gamma m} \, \Pi \right) + \frac{\partial}{\partial \mathbf{p}} \cdot \left(-\alpha \, \frac{\mathbf{p}}{\gamma} \, \Pi \right) = D \frac{\partial^2 \Pi}{\partial \mathbf{p}^2}. \tag{IV.3}$$

We are now tempted to impose as equilibrium distribution the relativistic analogous of the Maxwell-Boltzmann distribution, the **Jüttner distribution** (see [24]):

$$\Pi_J(\mathbf{p}) = \frac{1}{4\pi m^3 c^3} \frac{1}{Q^2 K_2(1/Q^2)} \; exp\left(-\frac{\gamma}{Q^2} \right), \tag{IV.4}$$

where $Q^2 = \frac{k_B T_e}{mc^2}$ is the quotient between the thermic energy and the mass energy of a diffusing particle, and K_2 is the second order modified Hankel function. For $Q^2 \ll 1$ the Jüttner distribution reduces to the familiar Maxwell-Bolzmann distribution. Once we impose the Jüttner distribution as equilibrium distribution we again recover the expression of the fluctuation distribution theorem (III.21).

With this we completely defined the ROUP in the inertial coordinate system $\mathcal{R}_{eq}$ in which the fluid is globally at rest, and implicitly in any other inertial coordinate system, via Lorentz transform.

If we want to give the ROUP in any other inertial coordinate system than $\mathcal{R}_{eq}$, we are clearly bound to introduce a vector describing the velocity of the surrounding fluid. It turns out that a description of the ROUP equivalent to equations (IV.1) (and to the Kolmogorov equation (IV.3)) is given by the following system of manifestly covariant equations:

$$\begin{cases} \dfrac{dx^\mu}{ds} = u^\mu \\ \dfrac{dp^\mu}{ds} = F^\mu + \xi^\mu, \end{cases} \tag{IV.5}$$

where s is the proper distance along the world line of the particle, indices denoted by Greek letters run from 0 to 3, and the chosen signature of the space-time metric is $(+,-,-,-)$. The 4-vector ξ^μ is the stochastic part of the 4-force acting on the particle, and F^μ is the deterministic part of the 4-force given by the expression:

$$F^\mu = -m\lambda^\mu_\nu (u^\nu - U^\nu) + m\lambda^\alpha_\beta u_\alpha (u^\beta - U^\beta) u^\mu. \tag{IV.6}$$

Here the time-like 4-vector U^μ represents the surrounding fluid's (local) 4-velocity, and the second rank tensor λ, which *a priori* depends on the thermodynamic state of the surrounding fluid and both velocities u^μ and U^μ, generalizes the usual friction-coefficient. We note that the deterministic 4-force is by construction orthogonal to the

4-velocity of the particle, $u_\mu F^\mu = 0$, so that the condition $u_\mu u^\mu = 1$ is not violated by the motion. If we assume that the fluid is isotropic the tensor λ takes the form:

$$\lambda^\mu_\nu = \chi U^\mu U_\nu + \alpha/\gamma^2(\delta^\mu_\nu - U^\mu U_\nu), \tag{IV.7}$$

with the two scalars χ and α. As a matter of fact, assuming the metric to be Minkowskian and that the surrounding fluid is in an equilibrium state, one finds immediately that, in the rest frame of the fluid $\mathcal{R}_{eq}$ where $U = (1, \mathbf{0})$, equation (IV.7) gives:

$$\lambda^\mu_\nu = \begin{pmatrix} \chi & 0 & 0 & 0 \\ 0 & \alpha/\gamma^2 & 0 & 0 \\ 0 & 0 & \alpha/\gamma^2 & 0 \\ 0 & 0 & 0 & \alpha/\gamma^2 \end{pmatrix}, \tag{IV.8}$$

which does not particularizes any spatial direction. In reference [8], the coefficient χ was chosen to be equal zero (see the discussion in Section 3.1 of reference [8]).

The random part of the force, ξ^μ , is characterized by the fact that its spatial part is the centered Gaussian white noise. In the coordinate system $\mathcal{R}_{eq}$ it is thus equal to $\sqrt{2D}d\mathbf{w}/dt$, where $\mathbf{w}$ is the 3-dimensional Wiener process.

It is now the case to introduce a manifestly covariant formalism also for the Kolmogorov formulation of the ROUP as it was done in references [1] and [7].

IV.2 Covariant Kolmogorov equation for the ROUP

In relativistic statistical physics it has become customary to introduce an extended 8-dimensional phase-space, which is essentially the Cartesian product of the space-time manifold $\mathcal{M}$ and of a corresponding extended 4-dimensional momentum-space, obtained by formally treating the four momentum components as independent variables. Actually the extended phase-space can be identified with the co-tangent bundle to the space-time manifold $T^*\mathcal{M} = \{(x^\mu, p_\mu)\}$ (the distinction between tangent and co-tangent bundle is just academic in special relativity, where we have to do with a flat metric, but it is not the case in general; moreover, some considerations (see for example reference [11]) makes the choice of the co-tangent bundle more natural). A new unphysical distribution function is then introduced on this extended phase-space. Every calculation is then carried out with this distribution function and the physical relevant results are then recovered by restricting every equation to the mass-shell, essentially by a convolution with a Dirac-delta function. This sort of formalism is more elegant, treating time as a space-time independent coordinate with its associated independent momentum-coordinate, and it generally sensibly simplifies calculations. Moreover, the use of a manifestly covariant formalism is the first step of an extension of the ROUP in the context of general relativity.

Now, the three spatial momentum-components can naturally take any real value. However, the range of variation one should choose for the zeroth momentum-component treated as an independent variable is not obvious. For the case of the ROUP the

choice of the entire real axis is actually not advisable, because in that case the coefficients characterizing the process are not always well-defined. It turns out that a natural choice for the subset of $\mathbb{R}^4$ in which the variation of the 4-vector p should be restrained is the "half-space" $\mathcal{P} = \{p \cdot U > 0 \mid p \in \mathbb{R}^4\}$, where U stands for the 4-velocity of the surrounding fluid. In any given reference frame the condition $p \cdot U > 0$ can be transcribed in terms of the zeroth component of p as $p_0 > \frac{\mathbf{pU}}{U^0}$. In the coordinate system $\mathcal{R}_{eq}$, in which the surrounding fluid is at rest, U reads $U^\mu = (1, 0, 0, 0)$ and the condition for the zeroth component of $p \in \mathcal{P}$ therefore reads $p_0 > 0$. This, together with the fact that in $\mathcal{R}_{eq}$ the 4-vectors p on mass-shell are described by $p \cdot U = mc\gamma(\mathbf{p})$, makes sure that the mass-shell is included in $\mathcal{P}$.

If we now have a Lorentz invariant distribution function f on the cotangent bundle of the space-time manifold $T^*\mathcal{M}$ we recover the physical distribution function Π with a restriction on the mass-shell:

$$\Pi(t, \mathbf{x}, \mathbf{p}) = \int_{\mathcal{P}} f(t, \mathbf{x}, p_0, \mathbf{p}) \delta(p_0 - mc\gamma(\mathbf{p})) dp_0. \tag{IV.9}$$

The Kolmogorov equation (IV.3) can then be rewritten as

$$\int_{\mathcal{P}} \mathcal{L}(f) \delta(p_0 - mc\gamma(\mathbf{p})) \frac{dp_0}{p_0} = 0, \tag{IV.10}$$

where the differential operator $\mathcal{L}$ is defined by [1]

$$\mathcal{L}(f) = \partial_\mu(p^\mu f) + \partial_p^\mu(mcF^\mu f) + DK^{\alpha\mu\beta\nu}\partial_p^\mu\left(\frac{p_\alpha p_\beta}{p \cdot U}\partial_p^\nu f\right). \tag{IV.11}$$

Here we introduced the abbreviation $\partial_p^\mu = \frac{\partial}{\partial p_\mu}$ for the partial derivative with respect to an arbitrary component of the momentum p, and $\partial\mu$ for the partial derivative with respect to an arbitrary space-time component. The tensor K is defined by

$$K^{\alpha\mu\beta\nu} = U^\alpha U^\beta \Delta^{\mu\nu} + U^\mu U^\nu \Delta^{\alpha\beta} - U^\alpha U^\nu \Delta^{\mu\beta} - U^\mu U^\beta \Delta^{\alpha\nu}, \tag{IV.12}$$

where Δ is the projector unto the subspace of momentum 4-space orthogonal to U:

$$\Delta^{\mu\nu} = \eta^{\mu\nu} - U^\mu U^\nu, \tag{IV.13}$$

the tensor η being the flat metric tensor of special relativity $\eta^{\mu\nu} = diag(1, -1, -1, -1)$. The deterministic 4-force F^μ in equation (IV.11) is the one already defined in equation (IV.6):

$$F^\mu = -\lambda^{\mu\nu} p_\nu \frac{p^2}{m^2c^2} + \lambda^{\alpha\beta}\frac{p_\alpha p_\beta}{m^2c^2}\, p^\mu, \tag{IV.14}$$

with

$$\lambda^{\mu\nu} = \frac{m^2c^2\alpha/\gamma^2}{(p \cdot U)^2}\, \Delta^{\mu\nu}. \tag{IV.15}$$

From this last equation we effectively see that $\mathcal{P}$ is the largest domain in which all coefficients of the manifestly covariant Kolmogorov equation are defined and regular.

The manifestly covariant special relativistic Kolmogorov equation is now simply $\mathcal{L}(f) = 0$. It can serve as a manifestly covariant definition of the special relativistic Ornstein-Uhlenbeck process, because its restriction to the mass-shell is, in $\mathcal{R}_{eq}$, identical to equation (IV.3), which itself fully defines the process in all Lorentz frames.

IV.3 Conditional entropy 4-current and stationary equilibrium distribution

In Section II.3 we saw the definition of the conditional entropy of a distribution function f with respect to a distribution function g. Following the traditional relativistic theories of continuous media it is natural to extend that definition introducing also an entropy 4-current for the ROUP. Using the formalism introduced in the precedent sections it is easier to proceed in a manifestly covariant manner.

Let us first of all seek for a time and position independent solution of the Kolmogorov equation. This will be a stationary equilibrium distribution function in phase-space which will represent thermodynamic equilibrium.

It can be shown (see reference [1]) that the Jüttner distribution satisfies these requirements. Here we write it in a manifestly covariant manner:

$$f_*(p) = \frac{1}{4\pi(mc)^3} \frac{\frac{mc^2}{k_B T_e}}{K_2(\frac{mc^2}{k_B T_e})} \; e^{-\frac{c}{k_B T_e}(p \cdot U)}. \tag{IV.16}$$

Now, we want to construct an entropy current starting from an entropy density, in the same way in which we usually construct a particle current from a particle density. We recall that the spatial particle density and its associated 3-current are defined respectively by:

$$n(x) = \int_{\mathbb{R}^4} f(x,p)\delta(p_0 - mc\gamma(\mathbf{p}))d^4p \tag{IV.17}$$

and

$$\mathbf{j}(x) = \int_{\mathbb{R}^4} \frac{\mathbf{p}}{p^0} \; f(x,p)\delta(p_0 - mc\gamma(\mathbf{p}))d^4p. \tag{IV.18}$$

In manifestly covariant relativistic kinetic theory it customary to combine these two quantities in a unique mathematical object defining a particle 4-current [5]. Thanks to the well-known properties of the Dirac δ function we can write:

$$\delta(p^2 - m^2c^2) = \frac{1}{2p^0}\big[\delta(p_0 - mc\gamma(\mathbf{p})) + \delta(p_0 - mc\gamma(\mathbf{p}))\big], \tag{IV.19}$$

and thus

$$2p^0 \; \theta(p_0)\delta(p^2 - m^2c^2) = \delta(p_0 - mc\gamma(\mathbf{p})). \tag{IV.20}$$

This allows us to transcribe the definitions of the spatial particle density and the particle 3-current, equations (IV.17) and (IV.18) respectively as:

$$n(x) = 2\int_{\mathbb{R}^4} p^0 \; f(x,p)\theta(p_0)\delta(p^2 - m^2c^2)d^4p \tag{IV.21}$$

and

$$\mathbf{j}(x) = 2\int_{\mathbb{R}^4} \mathbf{p} \; f(x,p)\theta(p_0)\delta(p^2 - m^2c^2)d^4p, \tag{IV.22}$$

that is, as the components of the current 4-vector defined as

$$j^{\mu}(x) = 2\int_{\mathbb{R}^4} p^{\mu} f(x,p)\theta(p_0)\delta(p^2 - m^2c^2)d^4p. \tag{IV.23}$$

Following this example the authors of reference [1] defined the entropy 4-current in the following way:

$$S^{\mu}_{f|g}(x) = -\int_{\mathbb{R}^4} p^{\mu} f(x,p) \ln\left(\frac{f(x,p)}{g(x,p)}\right) \theta(p_0)\delta(p^2 - m^2c^2)d^4p. \tag{IV.24}$$

Chapter V

The ROUP in the hydrodynamic limit

In this chapter we will show how to obtain, from the exact transport equation for the ROUP, an approximated simplified equation describing the diffusion process in the large scale limits. In the equilibrium reference frame of the ROUP, this equation surprisingly turns out to be the traditional diffusion equation. This is apparently contradictory, because, has we explained in one of the previous chapters, the parabolical structure of the usual diffusion equation would allow a signal propagation with unbounded speed. We will show how to solve this paradox and how to reconciliate the whole with the principles of Einstein relativity. We can already anticipate that the hydrodynamic limit implicitly assumes that we are considering velocities which are very small (compared to the speed of light). Loosely speaking, unbounded velocities simply means velocities that are greater than those considered in this regime. The whole discussion teated here is mainly based on references [6] and [9].

V.1 The ROUP in the large-scale limit

The different scales of the problem

In this chapter we will consider the case in which the fluid surrounding the diffusing particles is in thermodynamical equilibrium with temperature T_{eq}, and the whole discussion will be held in the coordinate sytem $\mathcal{R}_{eq}$ where the fluid is globally at rest.

We want to take a look at the scales which characterize the ROUP in this reference frame, that is to say the characteristic dimensions which occur in the Kolmogorov

transport equation (IV.3):

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\frac{\mathbf{p}}{\gamma m} \, \Pi \right) + \frac{\partial}{\partial \mathbf{p}} \cdot \left(-\alpha \, \frac{\mathbf{p}}{\gamma} \, \Pi \right) = D \frac{\partial^2 \Pi}{\partial \mathbf{p}^2}. \tag{V.1}$$

The factor α is the only time scale of the problem. Physically, $1/\alpha$ represents a characteristic microscopic relaxation time of the system. In the frame of statistical physics we could say that $1/\alpha$ is the mean time between two collisions of the diffusing particle with some surrounding fluid molecules.

A characteristic energy scale is surely given by the rest energy of the diffusing particles, $\epsilon_m = mc^2$. Another energy scale is defined by the thermic energy associated to the equilibrium energy, $\epsilon_{eq} = k_b T_{eq}$. Because of the fluctuation-dissipation theorem we have the following relation among ϵ_{eq}, α, D, and m:

$$\epsilon_{eq} = \frac{D}{m\alpha}. \tag{V.2}$$

Each of the energy scales ϵ_m and ϵ_{eq}, combined with the time scale $1/\alpha$, defines a length scale:

$$\lambda_m = \sqrt{\frac{\epsilon_m}{m\alpha^2}} = \frac{c}{\alpha}, \tag{V.3}$$

and

$$\lambda_{eq} = \sqrt{\frac{\epsilon_{eq}}{m\alpha^2}} = \frac{1}{m\alpha} \sqrt{\frac{D}{\alpha}}. \tag{V.4}$$

These two length scales are clearly bound by the relation:

$$\frac{\lambda_{eq}}{\lambda_m} = \sqrt{\frac{k_B T_{eq}}{mc^2}} = Q, \tag{V.5}$$

where the parameter Q, which was already introduced previously, measures the importance of the relativistic effects at the equilibrium temperature T_{eq}. In fact Q is zero in the galilean limit, where the only characteristic length is λ_{eq}, and it is infinite in the so-called ultra-relativistic limit, where the only remaining length scale is λ_m, λ_{eq} being infinite

The quantity λ_m represents, in microscopical physical terms, the distance covered between two collisions by a particle whose speed is c. While λ_{eq} is the distance covered between two collisions by a particle whose speed is the thermal speed $\sqrt{k_B T_{eq}/m}$.

Definition of 'large-scale'

The transport equation (V.1) gives an exact description of the ROUP at any scale. We could however be interested in the macroscopic behaviour of the system, hoping that a macroscopic approximated description will be sensibly simpler than an exact microscopic one.

To describe the system at a macroscopic scale, means to consider only the solutions to the transport equation which present a slow temporal and spatial variation with respect to the microscopic time and length scales.

The only characteristic microscopic time we have is $1/\alpha$. This naturally suggests to introduce the dimensionless time variable

$$T = \alpha t. \tag{V.6}$$

We say that a solution Π to equation (V.1) varies slowly with respect to the only time scale $1/\alpha$ of the problem if it is subjected to the following relation:

$$\eta = \left\| \frac{\partial_T \Pi}{\Pi} \right\|_\infty \equiv \max_{(t,\mathbf{x},\mathbf{p})} \left(\left| \frac{\partial_T \Pi}{\Pi} \right| \right), \tag{V.7}$$

where $\eta \ll 1$ is a "small parameter", and $|\cdot|$ can be for instance the Euclid norm.

Similarly we introduce the dimensionless spatial variable

$$\mathbf{X} = \frac{1}{\lambda_{eq}} \mathbf{x} \tag{V.8}$$

(λ_m would not be a good choice as characteristic length, since it goes to infinity in galilean regime). A solution Π with slow spatial variation is one subjected to the relation

$$\epsilon = \left\| \frac{\partial_{\mathbf{X}} \Pi}{\Pi} \right\|_\infty, \tag{V.9}$$

with another small parameter $\epsilon \ll 1$.

It is now advisable to introduce a dimensionless variable also for momentum. So let us define $\mathbf{P}$ by

$$\mathbf{P} = \sqrt{\frac{\alpha}{D}}\, \mathbf{p}. \tag{V.10}$$

The relativistic transport equation can now be cast in the form:

$$\frac{\partial \Pi}{\partial T} + \frac{\partial}{\partial \mathbf{X}} \cdot \left(\frac{\mathbf{P}}{\gamma(\mathbf{P})}\, \Pi \right) + \frac{\partial}{\partial \mathbf{P}} \cdot \left(-\frac{\mathbf{P}}{\gamma(\mathbf{P})}\, \Pi \right) = D \frac{\partial^2 \Pi}{\partial \mathbf{P}^2}, \tag{V.11}$$

where Π is seen as a function of T, $\mathbf{X}$, and $\mathbf{P}$, and where the Lorentz factor depends on $\mathbf{P}$ in the simple following way:

$$\mathbf{P} = \sqrt{1 + Q^2 \mathbf{P}^2}. \tag{V.12}$$

The Chapmann-Enskog expansion

The physical idea behind the Chapman-Enskog expansion relies on the assumption that, after a so-called relaxation phase in which a given distribution function Π varies on microscopic time and length scales, it will reach a slow time and spatial evolution phase. We can imagine that, waiting long enough, the diffusing particle will "thermalize" with the surrounding fluid and his probability distribution in momentum space

will attain a Maxwell-Boltzmann distribution corresponding to the equilibrium temperature T_{eq}. It therefore seems natural to introduce the notion of local equilibrium, described by the distribution function

$$\Pi_{loc}(t,\mathbf{x},\mathbf{p}) = \frac{n(t,\mathbf{x})}{4\pi m^3 c^3}\frac{1}{Q^2 K_2(1/Q^2)}\exp\left(-\frac{\gamma mc^2}{k_B T_{eq}}\right), \tag{V.13}$$

completely defined by giving the density field $n(t,\mathbf{x})$. We should also postulate that, during its slow evolution phase, the true solution of the transport equation Π is not very different from the local equilibrium distribution function Π_{loc} defined by the density particle $n(t,\mathbf{x})$ associated to Π. Saying that the two distribution functions are "not very different" we mathematically mean that we can expand the solution Π around Π_{loc} in terms of a small parameter ϵ' in the form:

$$\Pi(t,\mathbf{x},\mathbf{p}) = \Pi_{loc}(t,\mathbf{x},\mathbf{p}) + \sum_{k=1}^{\infty}\epsilon'^k\ \Pi_k(t,\mathbf{x},\mathbf{p}). \tag{V.14}$$

In general Π_{loc} is not a solution of the transport equation, and the fact to assume that it is a good approximation of the real solution is equivalent to suppose that such an exact solution Π is completely determined by the particle density $n(t,\mathbf{x})$, which is the only variable field in the definition of Π_{loc}.

At this point a slowly variable solution of the transport equation seems to depend only on the three small parameters η, ϵ and ϵ'. However, taking the moments of the Kolmogorov equation, shows that these parameters are not independent and must obey the following relations:

$$\eta = \epsilon^2 \tag{V.15}$$

and

$$\epsilon' = \epsilon. \tag{V.16}$$

It is then possible to solve the transport equation to any order, i.e. to find an expression for Π_k for any k, under the condition that the particle density $n(t,\mathbf{x})$ respects a solubility condition in the form of a differential equation. It turns out that this solubility condition is the same to any order [9], but even more surprising is the observation that this equation is the diffusion equation

$$\frac{\partial n}{\partial t} = \chi\frac{\partial^2 n}{\partial \mathbf{x}^2}, \tag{V.17}$$

where the coefficient χ is given by $\chi = \lambda_{eq}^2\alpha$. The authors of [9] even showed that the found solution Π verifies that same diffusion equation as the spatial density $n(t,\mathbf{x})$. This means that this equation is the wanted large scale description of the ROUP in his preferred reference frame.

V.2 Parabolic equations and Einstein relativity principle

The case of the ROUP

It is quite puzzling that the relativistic stochastic system of the ROUP can lead to the parabolic equation (V.17), which allows a propagation of matter at unbounded velocities. To better investigate this apparent paradox we recall that a solution $n(t, \mathbf{x})$ to the diffusion equation (V.17) can be represented as

$$n(t, \mathbf{x}) = \int_{\mathcal{R}^3} G(t, \mathbf{x} - \mathbf{x}')\ n(0, \mathbf{x}')\ d^3x', \tag{V.18}$$

where the Green function G associated to the problem is given by

$$G(t, \mathbf{x}) = \frac{1}{(4\pi\chi t)^{3/2}}\ \exp\left(-\frac{\mathbf{x}^2}{4\chi t}\right). \tag{V.19}$$

In order to get some insights on the origins of the paradox, we take a closer look at how the "hydrodynamic" scalings involved in deriving equation (V.17) from Kolmogorov equation work on $G(t, \mathbf{x})$. Applying the spatial scaling to G gives:

$$\left\|\frac{\partial_{\mathbf{X}} G}{G}\right\|_\infty = \mathcal{O}(\epsilon) \quad \Rightarrow \quad \left\|\frac{\mathbf{x}}{t}\right\|_\infty = \alpha\lambda_{eq}\mathcal{O}(\epsilon), \tag{V.20}$$

where the Landau notation $\mathcal{O}(\epsilon)$ denotes a term of order ϵ.

We now introduce the norm $\mathcal{N}(f)$ of any function $f(T, \mathbf{X}, \mathbf{P})$ with the definition

$$\mathcal{N}(f) = \max_{(T,\mathbf{X})} \int_{-\infty}^{\infty} |f|\, d^3\mathbf{P}. \tag{V.21}$$

This allows us to formalize the requirement that each term of the ϵ-expansion of Π is "small" with respect to previous one, i.e. $\mathcal{N}(\epsilon\Pi_1) \ll \mathcal{N}(\Pi_0)$ for the first terms. Inserting the expression that we can obtain for Π_0 and Π_1 (see reference [9]) we get the relation:

$$\epsilon Q \ll h(Q), \tag{V.22}$$

where $h(Q)$ is given by:

$$h(Q) = \frac{\exp(Q^{-2})K_1(Q^{-2})}{1 + Q^2}, \tag{V.23}$$

which is bounded by 1. This allows us to deduce the expression:

$$\epsilon Q \ll 1, \tag{V.24}$$

which in turn implies via equations (V.5) and (V.20):

$$\left\|\frac{\mathbf{x}}{t}\right\|_\infty \ll c. \tag{V.25}$$

These restrictions (V.20) and (V.25) clearly imply that the diffusion equation is a convenable large scale description of the ROUP only in the space-time domain where

the mean velocity of the diffusing particle is much smaller than the thermal speed $\alpha\lambda_{eq}$ and the light speed c. In this space-time domain the diffusion equation is clearly not in contradiction with Einstein special relativity. Outside this domain the diffusion equation predicts an acausal behaviour, but the conditions allowing the derivation of this equation from the transport equation of the ROUP are no longer valid, meaning that the diffusion equation is no longer a good description of the ROUP. The apparent paradox is therefore solved. Let us finally note that in the case of the galilean Ornstein-Uhlenbeck process, the restriction (V.25) is useless, because it is trivially satisfied, c being "infinite". We are only left with the restriction given by equation (V.20), which proves that, even in the galilean case, the diffusion equation is a good approximation of the Ornstein-Uhlenbeck process only in the limit of a small particles' velocity with respect to the thermal speed associated to the equilibrium temperature T_{eq}.

The general case

The conclusion we just met can give us some insights to the reasons of the difficulties in deriving a relativistic hydrodynamic theory. There are in fact many parallelisms between the study of the ROUP at large scales and the usual relativistic hydrodynamics. In usual hydrodynamics we almost always have, as a transport equation, the relativistic Boltzmann equation (instead of the Kolmogorov equation), and we introduce a local equilibrium function depending on a temperature and on a velocity field, in addition to the particle density n. The Chapman-Enskog procedure comes down to approximatively solving Boltzmann equation, searching for slowly variable solutions in the form of an expansion around this local equilibrium. This procedure, applied on the galilean Boltzmann equation gives back the usual Navier-Stokes hydrodynamics, as we explained in the first chapter. Implementing the Chapman-Enskog approach on the relativistic Boltzmann equation gives back the so-called first order theories, among which the most representative are the theories of Landau and Eckhart.

All these hydrodynamic equation are non-hyperbolic [21], presenting the same kind of paradox we found in the large scale approximation of the ROUP. And precisely the experience gained thanks to the analysis of the ROUP allows us to give an explanation and a solution to this (apparent) paradox [6]. We can in fact presume that the application of the Chapman-Enskog approach on the the relativistic Boltzmann equation leads to a system of partially parabolic equations, in the same way in which it lead to the parabolic diffusion equation when applied on the ROUP. There is not therefore any paradox, because the Chapman-Enskog approach itself imposes a restriction on the validity domain of these equations, and these are likely to be valid only in the space-time domain, where they are not in contradiction with special relativity, in exactly the same way in which the diffusion equation is only valid, as a description of the ROUP, when the particles' velocity is much smaller than c.

We could therefore believe illusory the quest for a relativistic dissipative macroscopic hydrodynamic theory, presuming that the state of a system in his local equilibrium reference frame varies on large space and time scales only in a space-time domain, where the macroscopic phenomena are non-relativistic.

Chapter VI

The General Relativistic Ornstein-Uhlenbeck Process

Having a manifestly covariant formulation of the ROUP, in the form of the manifestly covariant special relativistic Kolmogorov equation $\mathcal{L}(f) = 0$, an extension to the general relativistic framework is quite straight-forward. We only have to fix some technicalities to obtain a manifestly covariant general relativistic Kolmogorov equation.

First some words on the notation that will be adopted. Our Lorentzian metric tensor g will be chosen to have signature $(+,-,-,-)$. Indices running from 0 to 3 are indicated by Greek letters, whereas Latin letters will run from 1 to 3. Finally, $\det g$ will stand for the determinant of the coordinate basis components of the metric tensor g. Because of our signature choice the factor $(-\det g)$ will be positive.

VI.1 The measures on the extended phase-space

As it was said before, we choose the extended phase space to be the co-tangent bundle $T^*\mathcal{M}$ to the space-time manifold. We could have chosen also the tangent bundle, but experience in relativistic kinetic theory shows that the former choice is usually the most technically convenient solution [21]. Furthermore, in Hamiltonian mechanics momentum naturally appears as the conjugate degree of freedom to position; this means that choosing the position 4-vector to be contravariant, naturally induces a covariant momentum 4-vector.

These choices for the extended phase-space entail the following four-dimensional

volume measure in space-time:

$$\mathcal{D}^4x = \sqrt{-\det g}\; dx^4 = \sqrt{-\det g}\; \frac{1}{4!}\; \varepsilon_{\mu\nu\kappa\lambda}\; dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda, \tag{VI.1}$$

where $\varepsilon_{\mu\nu\kappa\lambda}$ is the completely antisymmetric symbol (see reference [25]).

As far as integration on the mass-shell is concerned, it is customary to introduce a measure which is defined over the whole momentum space but which enforces itself the mass-shell restriction. We thus define the following measure [21]:

$$\begin{aligned}\mathcal{D}^4p &= \theta(p_0)\delta(g^{\mu\nu}p_\mu p_\nu - m^2c^2)\frac{1}{\sqrt{-\det g}}\; \frac{1}{4!}\; \varepsilon^{\mu\nu\kappa\lambda}\; dp_\mu \wedge dp_\nu \wedge dp_\kappa \wedge dp_\lambda \\ &= \theta(p_0)\delta(g^{\mu\nu}p_\mu p_\nu - m^2c^2)\frac{1}{\sqrt{-\det g}}\; d^4p,\end{aligned} \tag{VI.2}$$

which is a pseudo-scalar.

VI.2 Manifestly covariant general relativistic Kolmogorov equation

The differential operator $\mathcal{L}$ as it was defined in equation (IV.11) is manifestly invariant under Lorentz transformation, but not under arbitrary coordinate change. The generalization of equation (IV.11) to curved space-times proposed in reference [7] is the following:

$$\mathcal{L}(f) = D_\mu(g^{\mu\nu}(x)p^\nu f) + \partial_p^\mu(mcF^\mu f) + DK^\alpha{}_\mu{}^\beta{}_\nu \partial_p^\mu \left(\frac{p_\alpha p_\beta}{p\cdot U}\partial_p^\nu f\right), \tag{VI.3}$$

where F and K given by equations (IV.12–IV.15), where of course η has to be substituted by its curved space-time counterpart g. The partial derivative with respect to an arbitrary position coordinate ∂_μ has been substituted by the differential operators D_μ, which is a sort of generalization of the Levi-Civita covariant derivative ∇_μ associated to the metric tensor g to the manifold represented by the extended phase-space $T^*\mathcal{M}$. The operator D_μ is defined as

$$D_\mu = \nabla_\mu + \Gamma^\alpha_{\mu\nu} p_\alpha \partial_p^\nu. \tag{VI.4}$$

The need to introduce this operator comes from the fact that the partial derivative ∂_μ is not a covariant operator. In fact if we consider an arbitrary field ϕ which transforms as a scalar under a coordinate change, i.e. $\phi'(x',p') = \phi(x,p)$, one has

$$\frac{\partial\phi'}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial\phi}{\partial x^\mu} + \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}\frac{\partial x^\mu}{\partial x^{\mu'}} p_{\nu'}\partial_p^\nu\phi, \tag{VI.5}$$

since

$$p_\nu = \frac{\partial x^{\nu'}}{\partial x^\nu} p_{\nu'}. \tag{VI.6}$$

That means that $\frac{\partial \phi}{\partial x^\mu}$ is not a covariant vector, and this simply because in passing from x to $x + dx$, the usual partial differentiation maintains the components of p constants, but, since in curved space-time the basis covectors in momentum space are themselves x dependent, this is not equivalent to maintaining the covector p itself constant. To maintain the covector (the real geometrical object) constant we have to parallel-transport p from x to $x + dx$, adding to the component in the coordinate basis at point $x + dx$ the amount $dp_\nu = \Gamma^\alpha_{\mu\nu} q_\alpha dx^\mu$, where Γ denotes the Christoffel symbols (see references [25, 42]).

On the other hand one has for a partial derivative with respect to an arbitrary momentum component:

$$\partial_p^{\mu'} \phi' = \frac{\partial x^{\mu'}}{\partial x^\mu} \partial_p^\mu \phi. \tag{VI.7}$$

This indicates that the operator ∂_p^μ indeed is a covariant operation, transforming scalar fields into tangent vector fields, and this simply because momentum space is a flat four-dimensional manifold, being the cotangent vector space in a precise point of the space-time manifold.

This is approximately how we can end up to the generalization (VI.3) of equation (IV.11).

We now want to bring the Kolmogorov equation $\mathcal{L}(f) = 0$ with $\mathcal{L}$ defined in equation (VI.3) in a more compact and practical form, which will facilitate further manipulations.

We start by inserting definition (VI.4) of the operator D_μ, developing the corresponding partial derivation with respect to p, and using the fact that the connection ∇ is the Levi-Civita connection associated to the space-time metric g, i.e. $\nabla g = 0$ (see reference [42]). Then we group all terms containing only first derivatives with respect to the various components of p:

$$\nabla_\mu(p^\mu f) + \Gamma^\alpha_{\mu\kappa} p_\alpha \partial_p^\kappa (g^{\mu\nu} p_\nu f) + \partial_p^\mu (mcF_\mu f) + DK^\alpha{}_\mu{}^\beta{}_\nu \partial_p^\mu \left(\frac{p_\alpha p_\beta}{p \cdot U} \partial_p^\nu f \right) = 0, \tag{VI.8}$$

$$\begin{aligned} &\partial_\mu(p^\mu f) + \partial_p^\kappa (\Gamma^\alpha_{\mu\kappa} g^{\mu\nu} p_\alpha p_\nu f) + \partial_p^\mu (mcF_\mu f) + DK^\alpha{}_\mu{}^\beta{}_\nu \partial_p^\mu \partial_p^\nu \left(\frac{p_\alpha p_\beta}{p \cdot U} f \right) \\ &\quad + DK^\alpha{}_\mu{}^\beta{}_\nu \partial_p^\mu \left[\partial_p^\nu \left(\frac{p_\alpha p_\beta}{p \cdot U} \right) f \right] = 0, \end{aligned} \tag{VI.9}$$

We now can write the Kolmogorov equation in the following compact form:

$$\partial_\mu(p^\mu f) + \partial_p^\kappa (\Gamma^\alpha_{\mu\kappa} g^{\mu\nu} p_\alpha p_\nu f) + \partial_p^\mu \left\{ I_\mu - \partial_p^\nu (J_{\mu\nu} f) \right\} = 0, \tag{VI.10}$$

where I and J are two tensor which are defined by

$$I_\mu = -DK^\alpha{}_\mu{}^\beta{}_\nu \, \partial_p^\nu \left(\frac{p_\alpha p_\beta}{p \cdot U} \right) + mcF_\mu, \tag{VI.11}$$

$$J_{\mu\nu} = -DK^\alpha{}_\mu{}^\beta{}_\nu \frac{p_\alpha p_\beta}{p \cdot U}. \tag{VI.12}$$

It is the case to stress that the general relativistic Kolmogorov equation involves the fluid surrounding the diffusing particle (through the 4-velocity U) as well as the gravitational field (through the metric tensor g). This means that the general relativistic Ornstein-Uhlenbeck process describes the stochastic motion of a diffusing particle interacting with both a given surrounding fluid in arbitrary motion and a given arbitrary gravitational field.

VI.3 Conditional entropy 4-current and stationary equilibrium distribution in curved space-time

For the conditional entropy 4-current of the ROUP in curved space-time it is clearly legitimate to propose the same manifestly covariant expression (IV.24) that was given for the ROUP in special relativity, with de adequate modifications due to the metric tensor g of general relativity (such the use of the measure defined in equation (VI.2) to integrate on the mass-shell)

$$S^{\mu}_{f|g}(x) = -\int_{\mathbb{R}^4} p^{\mu} f(x,p) \ln\left(\frac{f(x,p)}{g(x,p)}\right) \theta(p_0)\delta(g^{\mu\nu}p_\mu p_\nu - m^2c^2)\frac{1}{\sqrt{-\det g}}d^4p. \quad \text{(VI.13)}$$

In curved space-time it is however not in general possible to find a stationary equilibrium distribution function, as it was the case with the Jüttner distribution in the special relativistic case. It is in fact clear, to speak in rather rough terms, that there cannot always exist any time and position independent distribution function, given the fact that the metric tensor can depend on the space-time point.

We could however envisage the possibility to find a pseudo-stationary equilibrium distribution function, where the term "pseudo-stationary" itself should be clearly defined.

The fact that we do not have at our disposition any stationary equilibrium distribution function in curved space-time is in fact not that disturbing if our aim is to characterize the asymptotic behaviour of the entropy. It is true that the concept of conditional entropy is clear if we calculate the entropy conditional on the stationary equilibrium distribution function of the dynamical system, since in this case it simply represents the difference between the thermodynamic entropy and the maximal equilibrium entropy (see discussion at the end of Section II.3). Anyway it we can easily define the entropy of a distribution function f conditional on an arbitrary distribution function g, i.e. $H_c(f|g)$, as we saw in Section II.3, and the interpretation of this quantity will simply be the entropy difference between this two distributions. All this talk only to say that it makes sense to consider the quantity $H_c(f|g)$ even if g is not a stationary equilibrium distribution function. And this is indeed what we are obliged to do in curved space time, since there exist no stationary equilibrium distribution function in general.

Chapter VII

An H-theorem for the ROUP in curved space-time

We now analyse the behaviour of the entropy of the ROUP in general relativity. Our aim is to show that roughly the entropy increases with time. To be more specific, we will prove that in an arbitrary inertial frame, the conditional entropy of the process, calculated as 3-dimensional volume integral of the first component of the entropy current $S_{f|g}$, is a non-decreasing function of the time-coordinate in that reference frame. This can be proven by showing that the covariant divergence of the conditional entropy current is non-negative, i.e. $\nabla \cdot S_{f|g}(x) \geq 0$.

The careful reader may have noted that we evaluate the entropy of an arbitrary distribution function f conditional to another arbitrary distribution function g, which is not in general the stationary equilibrium distribution function of the system f_*. This is in accordance with the discussion done at the end of the last Section of the preceding Chapter. The quantity $H_c(f|g)$ will represent the entropy difference between the states determined by f and g. The H-theorem states that this (negative) entropy difference is supposed to be non-decreasing, and has therefore the tendency to attain its maximum value of zero, meaning that both states f and g have evolved to a state with the same content of entropy. If then there exists a unique thermodynamic equilibrium state (characterized by the fact that it presents a maximum entropy), then f and g are both supposed to evolve toward it.

The whole proof of the H-theorem for the ROUP in curved space-time did the object of a recent paper under publication [36] which is now rapidly presented.

The main idea of the proof is simply to demonstrate that the covariant divergence

of the conditional entropy current can be brought to the form:

$$\nabla \cdot S_{f|g}(x) = \int_{\mathcal{P}} J_{\mu\nu}(x,p)\ D^{\mu}D^{\nu}\mathcal{D}^{4}p.$$

At this point we only have to show that $J_{\mu\nu}$ is non-negative defined.

The main technical difficulty in accomplishing the first step is perhaps due to the fact that the 4-D volume measure $\mathcal{D}^4p$ given by equation (VI.2) depends on the space-time point x, because it contains the metric tensor $g_{\mu\nu}(x)$. This creates the difficulty that in calculating the covariant divergence of $S_{f|g}(x)$, we have to derive also the measure $\mathcal{D}^4p$, and in particular the Dirac δ function. The details of these calculations can be found in Appendix 3 of the paper [36]. We are now ready to present the paper in question.

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An H-theorem for the General Relativistic Ornstein-Uhlenbeck Process

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Abstract

We construct conditional entropy 4-currents for the general relativistic Ornstein-Uhlenbeck process and we prove that the 4-divergences of these currents are always non-negative. This H-theorem is then discussed in detail. In particular, the theorem is valid in any Lorentzian space-time, even those presenting well-known chronological violations.

Notations

In this article, c denotes the speed of light, and the signature of the space-time metric is $(+,-,-,-)$. Indices running from 0 to 3 are indicated by Greek letters. Latin letter indices run instead from 1 to 3. We also introduce the abbreviation $\partial_p^\mu = \frac{\partial}{\partial p_\mu}$ for the partial derivative with respect to an arbitrary component of the momentum p. This notation underlines the fact that this operator transforms as a contravariant vector. Similarly we will often write $\partial_\mu = \frac{\partial}{\partial x^\mu}$, but the latter operator naturally does *not* transform as a tensor. Finally, $\det g$ stands for the determinant of the coordinate basis components of the metric tensor g.

1 Introduction

In Galilean physics, the most common way to quantify the irreversibility of a phenomenon is to introduce an entropy *i.e.* a functional of the time-dependent thermodynamical state of the system which never decreases with time. In usual Galilean

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continuous media theories, the total entropy $\mathcal{S}$ can be written as the integral of an entropy density s over the volume occupied by the system [24]. One also introduces an entropy current $\mathbf{j}_s$ and, since entropy is by definition not generally conserved, the relation $\partial_t s + \nabla \cdot \mathbf{j}_s \geq 0$ holds for every evolution of the system.

Traditional relativistic hydrodynamics and kinetic theory deal with the problem in a completely similar manner. An entropy 4-current S is associated to the local thermodynamical state of the system [4,14,21]; the total entropy $\mathcal{S}(t_0)$ of the system at time-coordinate $t = t_0$ can be obtained by integrating S over the 3-D space-like submanifold $t = t_0$ and the entropy fluxes are obtained by integrating S over 2-D submanifolds of space-time. Since entropy is not generally conserved, the simple relation $\nabla \cdot S = \nabla_\mu S^\mu \geq 0$ holds for any evolution of the system.

Actually, given a system and its dynamics, any 4-vector field S of non-negative divergence which depends on the local thermodynamical state of the system can be considered as an entropy current. In particular, nothing precludes the possibility of associating more than one entropy current to a single local state of a system.

Let us illustrate this remark by considering two special cases of great physical and mathematical interest. Historically speaking, the first statistical theory of out-of-equilibrium systems is Boltzmann's model of dilute Galilean gases [4, 24, 13]. The local state of the system is encoded in the so-called one particle distribution function f, which obeys the traditional Boltzmann equation. A direct consequence of this equation is that a certain functional of the distribution function never decreases with time. Boltzmann denoted this functional by H and the result is therefore known as Boltzmann's H-theorem. To this day, H is the only-known functional of f that never decreases in time. This H-theorem has later on been extended to the relativistic generalization of Boltzmann's model of dilute gases [14]. Thus, the relativistic Boltzmann gas also admits an entropy (and an entropy current) and it seems that this entropy is unique.

The situation is drastically different for stochastic processes. Indeed, a theorem due to Voigt [22,26] states that, under very general conditions, a stochastic process admits an infinity of entropies: Let X be the variable whose time-evolution is governed by the stochastic process and let dX be a measure in X-space $\mathcal{X}$ (typically, dX is the Lebesgue measure if $\mathcal{X} \in \mathbb{R}^n$). Let now f and g be any two probability distribution functions solutions of the transport equation associated to the stochastic process. Then, the quantity

$$\mathcal{S}_{f|g}(t) = -\int_{\mathcal{X}} f(t,X) \ln\left(\frac{f(t,X)}{g(t,X)}\right) dX \qquad \text{(VII.1)}$$

is a never decreasing function of time and is called the conditional entropy of f with respect to g. Thus, to any given $f(t,\cdot)$ representing the state of the system at time t, one can associate as many entropies as there are different solutions g of the transport equation so, typically, an infinity. Naturally, if the function g_0 defined by $g_0(t,X) = 1$ for all t and X is a solution of the transport equation, the conditional entropy $\mathcal{S}_{f|g_0}$ of any distribution f with respect to g_0 coincides with the Boltzmann entropy of f.

The notion of conditional entropy corresponds to what is sometimes called the Kullback information. and we refer the reader to [3,18,19] for extensive discussions of this concept.

The application of Voigt's theorem to Galilean stochastic processes is of course straightforward and rather well-known, but its application to relativistic stochastic processes demands discussion. To be definite, we will now particularize our treatment to the ROUP, which is the first relativistic process to have been introduced in the literature [1,2,6,7,8].

Given a reference frame (chart) $\mathcal{R}$, the ROUP transcribes as a set of stochastic equations governing the evolution of the position and momentum of a diffusing particle as functions of the time coordinate t in $\mathcal{R}$. This set of equations is a stochastic process in the usual sense of the word, and Voigt's theorem ensures this process admits an infinity of conditional entropies. But, by construction, these entropies *a priori* depend on the reference frame $\mathcal{R}$ and the general theorem does not furnish any information about their tensorial status.

This question has been partly answered for the special relativistic Ornstein-Uhlenbeck process [1]. In flat space-time, the ROUP admits as invariant measure in p-space a Jüttner distribution J [16]; this distribution simply describes a special relativistic equilibrium at the temperature of the fluid surrounding the diffusing particle. It has been shown in [1] that this Jütnner distribution can be used to construct a 4-vector field of non-negative 4-divergence which can be interpreted as the conditional entropy current of f with respect to J.

The aim of the present article is to prove the existence of conditional entropy currents for the ROUP in curved space-time. The matter is organized as follows. Section 2 reviews some basic results pertaining to the ROUP in curved space-time with particular emphasis on the Kolmogorov equation associated to the process. It is also recalled here that, in a generic space-time, this equation does *not* admit any equilibrium stationary solution [6]. In particular, a general relativistic Jütnner distribution is not, generically, a solution of the Kolmogorov equation and, therefore, cannot be used to construct an entropy current in curved space-time. We therefore consider two arbitrary solutions f and g of the Kolmogorov equation and introduce in Section 3.1 a candidate for the conditional entropy current of f with respect to g. We then prove in Section 3.2 that the 4-divergence of this current is always non-negative. This is our main result and it constitutes an H-theorem for the ROUP in curved space-time. Note that the flat space-time version of this H-theorem is itself a new result because our previous work [1] only proved the existence of a single entropy current for the ROUP in flat space-time, *i.e.* the conditional entropy current of an arbitrary distribution f with respect to the Jüttner equilibrium distribution J. Finally, the new H-theorem and some of its possible extensions are discussed at length in Section 4. The Appendix recalls and, if necessary, proves some simple but important purely geometrical relations useful in deriving the H-theorem.

2 Basics on the ROUP in curved space-time

2.1 Kolmogorov equation

The general relativistic Ornstein-Uhlenbeck process can be viewed as a toy model for the diffusion of a point particle of non vanishing mass m interacting with both a fluid and a gravitational field. This process is best presented by its Kolmogorov equation in manifestly covariant form [6]. The extended phase-space is the eight-dimensional bundle cotangent to the space-time manifold with local coordinates, say (x^μ, p_ν), $(\mu,\nu) \in \{0,1,2,3\}^2$. At each point in space-time, the 4-D momentum space $\mathcal{P}$ is equipped with the 4-D volume measure:

$$\mathcal{D}^4 p = \theta(p_0)\delta(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}}\, d^4p, \tag{VII.2}$$

with $d^4p = dp_0 \wedge dp_1 \wedge dp_2 \wedge dp_3$. This measure behaves as a scalar with respect to arbitrary coordinate changes. Note that integrals over $\mathcal{P}$ defined by using (multiples of) $\mathcal{D}^4p$ as a measure are *de facto* restricted to the (generally position-dependent) mass-shell.

Let f be the probability distribution function in the extended phase-space of a particle diffusing in a surrounding fluid with normalized 4-velocity U. As shown in [6], f obeys a manifestly covariant Kolmogorov equation which can be written in the following compact form:

$$\partial_\mu(p^\mu f) = -\partial_p^\mu \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\}. \tag{VII.3}$$

The coefficients $\widetilde{\Gamma}_\mu$, which do not constitute a tensor, are defined by

$$\widetilde{\Gamma}_\mu = \Gamma^\lambda_{\mu\nu} g^{\kappa\nu} p_\kappa p_\lambda \tag{VII.4}$$

and

$$\mathcal{K}_\mu(f) = I_\mu f - \partial_p^\nu (J_{\mu\nu} f) \tag{VII.5}$$

with

$$I_\mu = -DK^\alpha{}_\mu{}^\beta{}_\nu\ \partial_p^\nu \left(\frac{p_\alpha p_\beta}{p\cdot U}\right) + mcF_\mu, \tag{VII.6}$$

$$J_{\mu\nu} = -DK^\alpha{}_\mu{}^\beta{}_\nu \frac{p_\alpha p_\beta}{p\cdot U}. \tag{VII.7}$$

The tensor K is independent of p. It depends on U and on the metric g, but only through the projector Δ on the orthogonal to U, which reads:

$$\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu. \tag{VII.8}$$

The explicit expression of K in terms of U and Δ is:

$$K^{\alpha\mu\beta\nu} = U^\alpha U^\beta \Delta^{\mu\nu} + U^\mu U^\nu \Delta^{\alpha\beta} - U^\alpha U^\nu \Delta^{\mu\beta} - U^\mu U^\beta \Delta^{\alpha\nu}. \tag{VII.9}$$

Finally, F represents the deterministic part of the force exerted by the fluid on the diffusing particle; its expression as a function of p and U reads

$$F_\mu = -\lambda_{\mu\nu} p^\nu \frac{p^2}{m^2c^2} + \lambda_{\alpha\beta}\frac{p^\alpha p^\beta}{m^2c^2} p_\mu, \tag{VII.10}$$

with

$$\lambda_{\mu\nu} = \frac{\alpha(mc)^2}{(p\cdot U)^2}\Delta_{\mu\nu}, \tag{VII.11}$$

$\alpha > 0$ being the friction coefficient (see [7]). Note that F is by construction orthogonal to p.

It has been shown in [6] that equation (VII.3) does not generically admit stationary solutions. In particular, a general relativistic Jüttner distribution cannot be used to construct in curved space-time a preferred conditional entropy current for the ROUP.

3 H-theorem for the ROUP in curved space-time

3.1 Definition of the conditional entropy currents

Given any two probability distribution functions f and g defined over the extended phase-space, a natural definition for the conditional entropy current of f with respect to g is:

$$S_{f|g}(x) = -\int_{\mathcal{P}} p\ f(x,p) \ln\left(\frac{f(x,p)}{g(x,p)}\right) \mathcal{D}^4 p. \tag{VII.12}$$

This definition is clearly the simplest generalization of equation (37) in ref. [1] to both an arbitrary reference distribution g and a possibly curved space-time background.

We will now prove that for all f and g solutions of the Kolmogorov equation (VII.3), the 4-divergence of $S_{f|g}$ is non-negative.

3.2 Proof of the H-theorem

The proof of the H-theorem for the General Relativistic Ornstein-Uhlenbeck process will be carried out in two steps.

3.2.1 Computation of the 4-divergence of the entropy current

Theorem 1. *For any f and g solutions of Kolmogorov equation*

$$\nabla\cdot S_{f|g}(x) = \int_{\mathcal{P}} J_{\mu\nu}(x,p)\ D^\mu[f/g]\, D^\nu[f/g]\, \mathcal{D}^4 p, \tag{VII.13}$$

where J is defined by equation (VII.7) and the functional D is given by:

$$D^\mu[f/g] = \partial_p^\mu \ln(f/g). \tag{VII.14}$$

Proof. The main idea behind the proof is to use Kolmogorov equation (VII.3) to convert all the spatial derivatives into derivatives with respect to momentum components. To do this we will deal with various integrals over $\mathcal{P}$ by integrating most of them by parts. This procedure generally leads to the appearance of so-called 'border terms'. Some of them trivially vanish if we suppose, as is customary in statistical physics, that phase-space distribution functions tend to zero sufficiently rapidly at infinity (in 4-D p-space). One is then left with border terms that are to be evaluated on the hyperplane $p\cdot U=0$. These also vanish for the following reason. Let us choose, at each point in space-time, an orthornormal basis (tetrad) (e_a), $a=0,1,2,3$ in the tangent space. Introducing the components p_a and U^a of p and U in this base, the normalization condition $U^2=1$ reads:

$$U^0=\sqrt{1+\sum_{i=1}^{3}(U^i)^2} \tag{VII.15}$$

so that:

$$U^0>\sqrt{\sum_{i=1}^{3}(U^i)^2}. \tag{VII.16}$$

The condition $p\cdot U=0$ becomes $p_0U^0+\sum_{i=1}^3 p_iU^i=0$; since $U^0>0$, this translates into:

$$p_0=-\frac{\sum_{i=1}^3 p_iU^i}{U^0}. \tag{VII.17}$$

It follows easily from (VII.16) and (VII.17) that $(p_0)^2<\sum_{i=1}^3(p_i)^2$ on the hyperplane $p\cdot U=0$. The Dirac δ distribution which enforces the on mass-shell restriction $p^2=m^2c^2$ therefore vanishes on the hyperplane $p\cdot U=0$, ensuring that the corresponding border terms disappear.

Let us now proceed with the proof of Theorem 1. Direct derivation of equation (VII.12) leads to:

$$\begin{aligned}\nabla_\kappa S^\kappa_{f|g}&=-\partial_\kappa\int_{\mathcal{P}}p^\kappa f\ln\left(\frac{f}{g}\right)\mathcal{D}^4p-\Gamma^\alpha_{\alpha\kappa}\int_{\mathcal{P}}p^\kappa f\ln\left(\frac{f}{g}\right)\mathcal{D}^4p\\&=\underbrace{-\int_{\mathcal{P}}\partial_\kappa(p^\kappa f)\ln\left(\frac{f}{g}\right)\mathcal{D}^4p}_{=\mathcal{A}_1}\ \underbrace{-\int_{\mathcal{P}}p^\kappa\left[(\partial_\kappa f)-\frac{f}{g}(\partial_\kappa g)\right]\mathcal{D}^4p}_{=\mathcal{A}_2}\\&\underbrace{-\int_{\mathcal{P}}p^\kappa f\ln\left(\frac{f}{g}\right)\partial_\kappa(\mathcal{D}^4p)}_{=\mathcal{A}_3}\ \underbrace{-\Gamma^\alpha_{\alpha\kappa}\int_{\mathcal{P}}p^\kappa f\ln\left(\frac{f}{g}\right)\mathcal{D}^4p}_{=\mathcal{A}_4}.\end{aligned} \tag{VII.18}$$

Using Kolmogorov equation (VII.3), integrating by parts, and inserting the defi-

nition of $\mathcal{K}_\mu(f)$ equation (VII.5) we obtain for $\mathcal{A}_1$:

$$\begin{aligned}\mathcal{A}_1 &= \int_{\mathcal{P}} \partial_p^\mu \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p \\ &= -\int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \left[I_\mu f - \partial_p^\nu (J_{\mu\nu} f)\right]\right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p \\ &\quad - \int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \ln\left(\frac{f}{g}\right) \partial_p^\mu (\mathcal{D}^4 p).\end{aligned} \tag{VII.19}$$

Let us now consider the term $\mathcal{A}_2$:

$$\begin{aligned}\mathcal{A}_2 &= -\int_{\mathcal{P}} p^\kappa \left[(\partial_\kappa f) - \frac{f}{g}(\partial_\kappa g)\right] \mathcal{D}^4 p \\ &= \underbrace{-\int_{\mathcal{P}} \partial_\kappa (p^\kappa f) \mathcal{D}^4 p}_{=\mathcal{B}_1} + \underbrace{\int_{\mathcal{P}} \partial_\kappa (p^\kappa g) \frac{f}{g}\, \mathcal{D}^4 p}_{=\mathcal{B}_2}.\end{aligned} \tag{VII.20}$$

Using again Kolmogorov equation (VII.3) and integrating by parts, we obtain for the term $\mathcal{B}_1$:

$$\mathcal{B}_1 = \int_{\mathcal{P}} \partial_p^\mu \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \mathcal{D}^4 p = -\int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \partial_p^\mu (\mathcal{D}^4 p), \tag{VII.21}$$

and for the term $\mathcal{B}_2$:

$$\begin{aligned}\mathcal{B}_2 &= -\int_{\mathcal{P}} \partial_p^\mu \left\{\widetilde{\Gamma}_\mu g + \mathcal{K}_\mu(g)\right\} \frac{f}{g}\, \mathcal{D}^4 p \\ &= \int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(g)\frac{f}{g}\right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p + \int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(g)\frac{f}{g}\right\} \partial_p^\mu (\mathcal{D}^4 p).\end{aligned} \tag{VII.22}$$

Summing (VII.21) and (VII.22) and inserting the definition of $\mathcal{K}_\mu(g)$ equation (VII.5) we obtain:

$$\begin{aligned}\mathcal{A}_2 &= \int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \left[I_\mu f - \partial_p^\nu (J_{\mu\nu} g)\frac{f}{g}\right]\right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p \\ &\quad + \int_{\mathcal{P}} \left\{\mathcal{K}_\mu(g)\frac{f}{g} - \mathcal{K}_\mu(f)\right\} \partial_p^\mu (\mathcal{D}^4 p).\end{aligned} \tag{VII.23}$$

Putting (VII.19) and (VII.23) together we get:

$$\begin{aligned}\mathcal{A}_1 + \mathcal{A}_2 &= \int_{\mathcal{P}} \left\{\partial_p^\nu (J_{\mu\nu} f) - \partial_p^\nu (J_{\mu\nu} g)\frac{f}{g}\right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p \\ &\quad - \int_{\mathcal{P}} \widetilde{\Gamma}_\mu f \ln\left(\frac{f}{g}\right) \partial_p^\mu (\mathcal{D}^4 p) \\ &\quad + \int_{\mathcal{P}} \left\{\mathcal{K}_\mu(g)\frac{f}{g} - \mathcal{K}_\mu(f)\left[1 + \ln\left(\frac{f}{g}\right)\right]\right\} \partial_p^\mu (\mathcal{D}^4 p).\end{aligned} \tag{VII.24}$$

The third integral on the right-hand side of equation (VII.24) contains two contributions and they both involve the contraction of the operator $\mathcal{K}$ with $\partial_p^\mu(\mathcal{D}^4 p)$. By equation (VII.49) in Appendix VII, this contraction is proportional to the contraction of $\mathcal{K}$ with p. By definitions (VII.5), (VII.6) and (VII.7), the action of this latter contraction on an arbitrary function h reads:

$$\begin{aligned} p^\mu \mathcal{K}_\mu(h) &= p^\mu \{ I_\mu h - \partial_p^\nu (J_{\mu\nu} h) \} \\ &= D K^\alpha{}_\mu{}^\beta{}_\nu p^\mu \frac{p_\alpha p_\beta}{p \cdot U} (\partial_p^\nu h) + mc p^\mu F_\mu h. \end{aligned} \tag{VII.25}$$

The tensor $K^{\alpha\mu\beta\nu}$ is antisymmetric upon exchange of the indices μ and α, entailing that $K^{\alpha\mu\beta\nu} p_\alpha p_\mu p_\beta = 0$; moreover, the deterministic 4-force F is orthogonal to the momentum p, i.e. $p^\mu F_\mu = 0$. Equation (VII.25) therefore simply reduces to:

$$p^\mu \mathcal{K}_\mu(h) = 0. \tag{VII.26}$$

The last integral in equation (VII.24) therefore disappears, and we can write:

$$\begin{aligned} \mathcal{A}_1 + \mathcal{A}_2 = \int_{\mathcal{P}} f \underbrace{\left\{ \frac{1}{f} \partial_p^\nu (J_{\mu\nu} f) - \frac{1}{g} \partial_p^\nu (J_{\mu\nu} g) \right\}}_{= J_{\mu\nu} D^\mu [f/g]} D^\mu [f/g] \, \mathcal{D}^4 p \\ - \Gamma^\nu_{\mu\kappa} \int_{\mathcal{P}} p^\kappa p_\nu f \ln \left(\frac{f}{g} \right) \partial_p^\mu (\mathcal{D}^4 p), \end{aligned} \tag{VII.27}$$

where we used definition (VII.14) of $D^\mu[\cdot]$ and definition (VII.4) of $\widetilde{\Gamma}_\mu$.

Let us now address the $\mathcal{A}_3$ contribution to equation (VII.18). Inserting the expression (VII.50) (from Appendix A.3) for $\partial_\kappa(\mathcal{D}^4 p)$, we have:

$$\begin{aligned} \mathcal{A}_3 &= - \int_{\mathcal{P}} p^\kappa f \ln \left(\frac{f}{g} \right) \partial_\kappa (\mathcal{D}^4 p) \\ &= \Gamma^\nu_{\kappa\mu} \int_{\mathcal{P}} p^\kappa p_\nu f \ln \left(\frac{f}{g} \right) \partial_p^\mu (\mathcal{D}^4 p) + \Gamma^\alpha_{\alpha\kappa} \int_{\mathcal{P}} p^\kappa f \ln \left(\frac{f}{g} \right) \mathcal{D}^4 p. \end{aligned} \tag{VII.28}$$

Inserting equations (VII.28) and (VII.27) in (VII.18), we obtain the wanted simple expression:

$$\nabla_\mu S^\mu_{f|g} = \int_{\mathcal{P}} J_{\mu\nu} \; D^\mu [f/g] \, D^\nu [f/g] \; \mathcal{D}^4 p. \tag{VII.29}$$

□

3.2.2 The 4-divergence of the entropy current is non-negative

We now state a second theorem, which, together with the previous one, will prove the H-theorem.

Theorem 2. *For any two arbitrary distributions f and g, the integrand in equation (VII.13) of Theorem 1 is non-negative, that is:*

$$J_{\mu\nu}\ D^{\mu}\left[f/g\right] D^{\nu}\left[f/g\right] \geq 0. \tag{VII.30}$$

Proof. Let us fix an arbitrary point x in space-time and choose as local reference frame $(\mathcal{R})$ at x the proper rest frame at x of the fluid surrounding the diffusing particle. By definition, in this reference frame, the components of the 4-velocity $U(x)$ of the fluid at x are simply $U^{\mu} = \frac{1}{\sqrt{g_{00}}}(1,0,0,0)$. Inserting these components into the definition (VII.7) for J, we get:

$$J^{00} = -\frac{D}{\sqrt{g_{00}}p_0} g^{ij} p_i p_j, \tag{VII.31}$$

$$J^{0i} = -D\left(\frac{1}{g_{00}}(p_0)^2 g^{0i} - \frac{1}{g_{00}} p_0 g^{i\alpha} p_{\alpha}\right)\frac{\sqrt{g_{00}}}{p_0} = \frac{D}{\sqrt{g_{00}}p_0} g^{ij} p_0 p_j, \tag{VII.32}$$

$$J^{ij} = -D\left(\frac{1}{g_{00}}(p_0)^2 g^{ij}\right)\frac{\sqrt{g_{00}}}{p_0} = -\frac{D}{\sqrt{g_{00}}p_0} g^{ij}(p_0)^2. \tag{VII.33}$$

We thus find:

$$\begin{aligned} J^{\mu\nu} D_{\mu} D_{\nu} &= J^{00} D_0 D_0 + 2J^{0i} D_0 D_i + J^{ij} D_i D_j \\ &= -\frac{D}{\sqrt{g_{00}}p_0}\left[g^{ij} p_i p_j (D_0)^2 - 2g^{ij} p_0 p_j D_0 D_i + g^{ij}(p_0)^2 D_i D_j\right] \\ &= -\frac{D}{\sqrt{g_{00}}p_0} \underbrace{\left[p_i D_0 - (p_0)^2 D_i\right]}_{=v_i} g^{ij} \underbrace{\left[p_j D_0 - (p_0)^2 D_j\right]}_{=v_j} \\ &= -\frac{D}{\sqrt{g_{00}}p_0} g^{ij} v_i v_j. \end{aligned} \tag{VII.34}$$

By Lemma 1 presented in Appendix VII, the right-hand side of this equation is non-negative, which proves Theorem 2. □

4 Discussion

This article has been focused on the General Relativistic Ornstein-Uhlenbeck process introduced in [6]; we have constructed a conditional entropy 4-current associated to any two arbitrary distributions solutions of Kolmogorov equation for the ROUP, and we have proven that the 4-divergence of this current is always non negative; this constitutes an H-theorem for the ROUP in curved space-time. It is a twofold generalization of the theorem introduced in [1]. First, the H-theorem proved in [1] concerns flat space-time only. Second, [1] does not deal with a conditional entropy 4-current associated to *two* arbitrary distributions, but only with the conditional entropy 4-current

of *one* arbitrary distribution with respect to the equilibrium distribution (invariant measure) of the ROUP in flat space-time. Let us note in this context that the ROUP does not generally admit an equilibrium distribution in curved space-time [6].

We would like now to comment on this new H-theorem. Let us first remark that the theorem is valid in any Lorentzian space-time and for any time-like field U representing the velocity of the fluid in which the particles diffuse. In particular, the theorem is even valid in space-times with closed time-like curves, as the Gödel universe or the extended Kerr black hole [12], and even if U is tangent to one of these closed time-like curves. The irreversibility measured by the local increase of the conditional entropy currents is entirely due to the Markovian character [11,23,25] of the ROUP and the remarkably general validity of the H-theorem proves that this irreversibility is in some sense stronger than all possible general relativistic chronological violations.

It should nevertheless be remarked that, as the Boltzmann-Gibbs entropy current associated to the relativistic Boltzmann equation, the conditional entropy 4-currents introduced in Section 3.1 are not necessarily time-like. And, even when they are time-like, their time-orientation in an orientable space-time generally depends on the point at which they are evaluated. Let us elaborate on this by first recalling the definition of the Boltzmann-Gibbs entropy current $S_{BG}[f]$ associated to a distribution f (see reference [14]):

$$S_{BG}[f](x) = -\int_{\mathcal{P}} p\ f \ln f\ \mathcal{D}^4 p. \tag{VII.35}$$

The normalization of f reads:

$$1 = \int_{\mathcal{T}_\Sigma} f\ d^3x \mathcal{D}^4 p, \tag{VII.36}$$

where Σ is an arbitrary space-like hypersurface of the space-time $\mathcal{M}$ and where $\mathcal{T}_\Sigma \subset T^*(\mathcal{M})$ is defined by

$$\mathcal{T}_\Sigma = \{(x,p) \in T^*(\mathcal{M}), x \in \Sigma\}. \tag{VII.37}$$

As a probability distribution, f is certainly non-negative; but f may take values both superior and inferior to unity. Therefore, nothing can be said on the sign of the function $f \ln f$ against which the time-like vector p is integrated in (VII.35). This entails that $S_{BG}[f](x)$ may be either time-like or space-like. Also note that the sign of the zeroth component of $S_{BG}[f](x)$ cannot be ascertained either; thus, even when time-like, the Boltzmann-Gibbs entropy current may be past as well as future oriented (in a time-orientable space-time).

Similarly, the sign of the function $f(x,p)\ln(f(x,p)/g(x,p))$ appearing in definition (VII.12) of the conditional entropy current $S_{f|g}(x)$ generally depends on p (and x) and $S_{f|g}(x)$ may therefore not be time-like. For the same reason, the sign of the zeroth component of $S_{f|g}(x)$ also generally depends on the point in space-time so that the conditional entropy currents, even when time-like, may not have a definite time-orientation (in a time-orientable space-time).

The Galilean limit deserves a particular discussion. The very notions of time-like and space-like vector-fields do not exist in this limit and only the time-orientation

of the conditional entropy currents should be addressed. In the Galilean limit, the zeroth component of $S_{f|g}(x)$ reads

$$s_{f|g}(t,\mathbf{x}) = -\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{p}) \ln\left(\frac{f(t,\mathbf{x},\mathbf{p})}{g(t,\mathbf{x},\mathbf{p})}\right) d^3p; \tag{VII.38}$$

note that this expression coincides with the conditional entropy density of the usual, non relativistic Ornstein-Uhlenbeck process [22]. A reasoning similar to the one presented in the preceding paragraph shows that this density may take positive as well as negative values. The time-orientation of the conditional entropy currents is therefore generally position-dependent, even in the Galilean regime.

However, in the Galilean limit, it surely makes sense to integrate $s_{f|g}(t,\mathbf{x})$ over the whole 3-D space to obtain the total (time-dependent) conditional entropy $\mathcal{S}(t)$ of f with respect to g and this quantity can be proven to be non positive. The proof [3,22] is based on the so-called Gibbs-Klein inequality [25]

$$F \ln F \geq F - 1, \tag{VII.39}$$

valid for any positive real number F and applied to $F(t,\mathbf{x},\mathbf{p}) = f(t,\mathbf{x},\mathbf{p})/g(t,\mathbf{x},\mathbf{p})$ (with the hypothesis that g does not vanish anywhere in $\mathbb{R}^3$). One has indeed:

$$\begin{aligned}\int_{\mathcal{V}} s_{f|g}(t,\mathbf{x}) d^3x &= -\int_{\mathcal{V}\times\mathbb{R}^3} f(t,\mathbf{x},\mathbf{p}) \ln\left(\frac{f(t,\mathbf{x},\mathbf{p})}{g(t,\mathbf{x},\mathbf{p})}\right) d^3x\; d^3p \\ &\leq \int_{\mathcal{V}\times\mathbb{R}^3} \left(f(t,\mathbf{x},\mathbf{p}) - g(t,\mathbf{x},\mathbf{p})\right) d^3x\; d^3p \\ &\leq 0. \end{aligned} \tag{VII.40}$$

This calculation can be extended formally to the special and general relativistic situations, but, since conditional entropy 4-currents are then not necessarily time-like, their integrals on space-like 3-D submanifolds may take positive or negative values. It is therefore far from clear that the concept of *total* conditional entropy makes sense in the relativistic regime. In particular, the relativistic H-theorem proved in this article should be primarily considered as a purely local result.

Thus, the conceptual status of the entropy currents introduced in Section 3.1 is in a certain sense similar to the status of the general relativistic black hole entropies [15,17,28,29]. Indeed, we have shown in this article that stochastic processes theory proves the existence of conditional entropy currents in curved space-time and permits their computation, exactly as quantum field theory and string theory both prove the existence of black-holes entropies and furnish the tools necessary for their computations. But the standard statistical interpretation of conditional entropy currents via their fluxes through 3-D space-like submanifolds is certainly not straightforward in curved space-time, as the usual interpretation of entropy and temperature via Gibbs canonical ensembles does not seem to extend smoothly to black hole thermodynamics [29].

It is our opinion that progress in interpreting the notion of entropy in curved space-time can best be achieved by studying specific examples in particular circumstances

where most results can be obtained by explicit or semi-explicit calculations. The ROUP is obviously an interesting tool for such computations and diffusion in space-times exhibiting naked or unnaked singularities should certainly be studied in detail.

Finally, it would naturally be most interesting to determine if H-theorems can also be proved for the two 'new' relativistic stochastic processes recently proposed as alternative models of relativistic diffusion in [9] and [10].

Appendix

A.1 General relations

A basic assumption of General Relativity is that the connection ∇ used in space-time is the Levi-Civita connection of the space-time metric g [27]. Given a coordinate basis, this translates into the following relation between the metric components $g_{\mu\nu}$ and the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$:

$$\partial_\kappa g_{\mu\nu} = \Gamma^{\alpha}_{\kappa\mu} g_{\alpha\nu} + \Gamma^{\alpha}_{\kappa\nu} g_{\mu\alpha}. \tag{VII.41}$$

Another equivalent form of (VII.41) is:

$$\partial_\kappa g^{\mu\nu} = -\Gamma^{\mu}_{\kappa\alpha} g^{\alpha\nu} - \Gamma^{\nu}_{\kappa\alpha} g^{\mu\alpha}. \tag{VII.42}$$

A direct consequence of equation (VII.42) is that, for any vector p:

$$(\partial_\kappa g^{\mu\nu}) p_\mu p_\nu = -\Gamma^{\mu}_{\kappa\alpha} p^\alpha p_\mu - \Gamma^{\nu}_{\kappa\alpha} p^\alpha p_\nu = -2\Gamma^{\nu}_{\kappa\mu} p_\nu p^\mu. \tag{VII.43}$$

Another useful relation reads [20]:

$$\partial_\kappa \det g = (\det g) g^{\mu\nu} \partial_\kappa g_{\mu\nu}. \tag{VII.44}$$

Using (VII.41), this translates into:

$$\partial_\kappa \det g = (\det g) g^{\mu\nu} 2\Gamma^{\alpha}_{\kappa\mu} g_{\alpha\nu} = 2(\det g)\Gamma^{\alpha}_{\kappa\alpha}. \tag{VII.45}$$

A.2 A useful lemma

Lemma 1. *Let (∂_μ) be a (local) coordinate basis of a Lorentzian space-time (with time-like ∂_0). Then, at any point x of space-time, the set of the six spatial components $g^{ij}(x)$ of the inverse metric tensor define a non-positive quadratic form. More precisely,*

$$g^{ij}(x) v_i v_j \leq 0 \text{ for all } (v_1, v_2, v_3) \in \mathbb{R}^3.^{3} \tag{VII.46}$$

Proof. Let x be a point in space-time and suppose there exists a set of three real numbers (v_1, v_2, v_3) such that $g^{ij}(x) v_i v_j > 0$. Define V, cotangent to the space-time manifold at x, by its components $V_0 = 0$, $V_1 = v_1$, $V_2 = v_2$, $V_3 = v_3$. The vector V is both time-like and orthogonal to ∂_0. The space cotangent to the space-time manifold at x therefore admits a time-like subspace of dimension at least two, which is impossible for a Lorentzian space-time. This proves the lemma. □

[3] See for example §84 of [20].

A.3 Derivatives of the volume measure in momentum-space

Let us now evaluate the partial derivatives of the volume measure $\mathcal{D}^4p$ with respect to both space-time coordinates and momentum components. The measure $\mathcal{D}^4p$ is defined by an expression which involves the product of a Heaviside function and a Dirac distribution. Direct derivation of this expression would lead to a product of Dirac distributions, which is not a well-defined mathematical object. To avoid this (at least formal) problem, we introduce a class of regular functions h_ϵ, which uniformly converge towards δ as ϵ tends to zero and write:

$$\begin{aligned}\partial_p^\mu\{\theta(p_0)\delta(p^2-m^2c^2)\} &= \lim_{\epsilon\to 0}\partial_p^\mu\{\theta(p_0)h_\epsilon(g^{\alpha\beta}p_\alpha p_\beta - m^2c^2)\}\\ &= \lim_{\epsilon\to 0}\{\delta(p_0)\delta_0^\mu h_\epsilon(g^{\alpha\beta}p_\alpha p_\beta - m^2c^2) + \theta(p_0)\partial_p^\mu[h_\epsilon(g^{\alpha\beta}p_\alpha p_\beta - m^2c^2)]\}\\ &= \lim_{\epsilon\to 0}\{\delta(p_0)\delta_0^\mu h_\epsilon(g^{ij}p_i p_j - m^2c^2) + \theta(p_0)2g^{\mu\nu}p_\nu h'_\epsilon(g^{\alpha\beta}p_\alpha p_\beta - m^2c^2)\}.\end{aligned} \tag{VII.47}$$

By the lemma proved in Section 3.2.2, $g^{ij}p_ip_j \leq 0$. The argument of h_ϵ in the last line of (VII.47) is therefore always strictly negative. The term involving h_ϵ thus disappears for $\epsilon \to 0$ and we are left with the result:

$$\partial_p^\mu\{\theta(p_0)\delta(p^2-m^2c^2)\} = 2p^\mu\theta(p_0)\delta'(p^2-m^2c^2). \tag{VII.48}$$

This equation leads directly to the following expression for the partial derivatives of $\mathcal{D}^4p$ with respect to momentum components:

$$\begin{aligned}\partial_p^\mu(\mathcal{D}^4p) &= \partial_p^\mu\left\{\theta(p_0)\delta(p^2-m^2c^2)\frac{1}{\sqrt{-\det g}}\right\}d^4p\\ &= 2p^\mu\theta(p_0)\delta'(p^2-m^2c^2)\frac{1}{\sqrt{-\det g}}d^4p.\end{aligned} \tag{VII.49}$$

Let us now focus on the derivatives of $\mathcal{D}^4p$ with respect to space-time coordinates. Using equations (VII.43), (VII.45) and (VII.49), we obtain:

$$\begin{aligned}\partial_\kappa(\mathcal{D}^4p) &= \partial_\kappa\left\{\theta(p_0)\delta(g^{\mu\nu}p_\mu p_\nu - m^2c^2)\frac{1}{\sqrt{-\det g}}\right\}d^4p\\ &= \theta(p_0)(\partial_\kappa g^{\mu\nu})p_\mu p_\nu\delta'(p^2-m^2c^2)\frac{1}{\sqrt{-\det g}}\ d^4p\\ &+ \theta(p_0)\delta(p^2-m^2c^2)\partial_\kappa\left(\frac{1}{\sqrt{-\det g}}\right)d^4p\\ &= -2\Gamma^\nu_{\kappa\mu}p_\nu p^\mu\theta(p_0)\delta'(p^2-m^2c^2)\frac{1}{\sqrt{-\det g}}\ d^4p\\ &- \theta(p_0)\delta(p^2-m^2c^2)\frac{1}{\sqrt{-\det g}}\frac{\partial_\kappa\det g}{2\det g}\ d^4p\\ &= -\Gamma^\nu_{\kappa\mu}p_\nu\partial_p^\mu(\mathcal{D}^4p) - \Gamma^\alpha_{\kappa\alpha}\mathcal{D}^4p.\end{aligned} \tag{VII.50}$$

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Concluding remarks

Relativistic stochastic processes like the ROUP are still a very actual research topic, both in mathematics and in physics. It even seems that a growing interest on this subject is spreading among the scientific community. For instance, the physicists P. Hänngi and J. Dunkel have recently published two articles [13, 14] on a relativistic version of the Brownian motion, that reduces to the standard Brownian motion in the Newtonian limit case, which is very similar in spirit to the construction put forward in 1997 by F. Debbasch, K. Mallick and J.P. Rivet [8].

On the other hand the mathematicians J. Franchi and Y. le Jan proposed an extended work on a relativistic diffusion process in a Schwarzschild geometry inspired by the founding papers of R.M. Dudley [12]. This construction has not a direct physical interpretation, but it is nonetheless interesting as a successful combination of the theory of stochastic processes with lorentzian differential geometry, giving back a diffusion process which is compatible with general relativity around a Schwarzschild black hole.

Also the physicists O. Oron and L.P. Horwitz recently wrote on the subject [32], but the purpose of this work was related to a relativistic generalization of Nelson stochastic mechanics [30] in the hope to find a covariant Brownian motion which would be associated with Parisi-Wu stochastic quantization [33]. Nelson [30] himself has pointed out that the formulation of his stochastic mechanics in the context of general relativity is an important open question, and the hope that was expressed in [32] is that "*the Riemannian metric spaces [...] which arise due to nontrivial correlations between fluctuations in space-time directions, could, in the framework of a covariant theory of Brownian motion, lead to spacetime pseudo-Riemannian metrics in the structure of diffusion and Schrödinger equations*". This is however admittedly a quite exotic issue and we do not feel capable of expressing a clear judgement on it.

Coming back to the special case of the ROUP, we feel it is the case to underline a noteworthy technical feature it shows up, namely the fact that, starting from a common gaussian white noise, it was possible to obtain a stochastic system which

relaxes to the non-trivial Jüttner distribution. This potentially opens up a research topic in probability theory, undeniably offering a new point of view on the good old gaussian distribution and on what it is possible to do with it.

Concluding, the structure of relativistic stochastic processes seems to be rich enough to justify their examination and explain the interest awaken in those who are involved in their study.

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