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Quantum Mechanics, Hidden Variables and Quantum Logic

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Contents

1 Introduction

In this text we will give a brief overview of quantum mechanics and its mathematical core, quantum probability theory, pointing out the differences between the main interpretations of quantum physics which arise. We will consider the so called 'orthodox' interpretation, the problems of measurement, definiteness and determinism entailed by it, and the attempts of solving this difficulties thanks to Hidden Variables theories. Then we will illustrate the constraints to this theories represented by the so called no-go theorems of Von Neumann, Bell and Kochen-Specker. This last important theorem finds its natural environment in the mathematical structures of partial Boolean algebras. That's why we will give a rapid introduction to this argument and its relations with Quantum Logic, before demonstrating the Kochen-Specker theorem.

2 Quantum probability and quantum mechanics

As we know the predictions of quantum mechanics are expressed in terms of expectations values of observables. This means that we have to do with *probability measures* over the Hilbert space of the states of a quantum system. On a mathematical point of view this can be stressed basing the entire foundation of QM on a probability theory: quantum probability theory.

In order to understand quantum probability theory we initially want to take a look to its classical counterpart, classical probability theory.

2.1 Classical probability

As we know (classical) probability theory allows us to assign to any element of a given set of events Ω, the sample space, a probability for its realization. If for example we take the simple events to be the possible results of rolling a six-sided die one time, the sample space is the set $\{1, 2, 3, 4, 5, 6\}$. We assign a probability to any element thanks to the *probability measure* p. For a laplacian six-sided die we may define: $p(\{i\}) = 1/6$, where $i = 1, \ldots, 6$.

Like in our example this probability on Ω is usually already known at the beginning and we want to express it on the algebra $\mathcal{F}_\Omega$ given by all possible logical combinations of the events. We represent logical combinations with the set-theoretic operations of intersection (which represents 'and'), union (which represents 'or'), and complement (which represents 'not'). Thus $\mathcal{F}_\Omega$ is obtained by Ω closing it under these operations. In the example of our six-sided die the event '2 or 3' is represented by $\{2\} \cup \{3\}$, which is $\{2, 3\}$, and so on.

Kolmogorov's axioms given below allow us to extend a probability measure p from Ω to the entire algebra $\mathcal{F}_\Omega$. In our example $p(\{2\} \cup \{3\}) = p(\{2\}) + p(\{3\}) = 1/6 + 1/6 = 1/3$.
Let's resume all this in the following scheme:

> *We identify a* ***classical probability*** *theory with an ordered triple* $(\Omega, \mathcal{F}_\Omega, p)$*, where:*
>
> Ω : *Sample space (of events)*
> $\mathcal{F}_\Omega$: *Algebra generated by* Ω
> *(closing it under: complement,* $\cap$*, and* $\cup$*)*
> p : *Kolmogorovian probability measure*
>
> ***Kolmogorov's axioms:***
>
> *(1)* $p(\emptyset) = 0$
> *(2)* $p(\neg F) = 1 - p(F)$
> *(3)* $p(F \cup F') = p(F) + p(F') - p(F \cap F')$.

2.2 Quantum probability

Quantum probability theory also begins with an ordered triple, $(\mathcal{H}, \mathsf{L}_{\mathcal{H}}, |\psi\rangle)$.

$\mathcal{H}$ is a separable Hilbert space, which is a (complete, complex) vector space with an inner product defined on it. Every one-dimensional subspace (or, equivalently, every normalized vector) in $\mathcal{H}$ corresponds to a simple event, so that $\mathcal{H}$ may be considered the sample space.

We generate an algebra of events, $\mathsf{L}_{\mathcal{H}}$, from $\mathcal{H}$ as follows. Take the set of all the one-dimensional subspaces of $\mathcal{H}$, and close it under the operations of span, intersection, and orthogonal complement. These operations correspond to the lattice-theoretic operations of join (denoted by '$\vee$'), meet (denoted by '$\wedge$'), and orthocomplement (denoted by '$^{\perp}$'), respectively, which leads some to interpret them as the quantum-mechanical representation of the logical operations of or, and, and, not. This in fact is the basis idea of Quantum logic, which will be discussed in section 4.

The algebra of quantum-mechanical events is denoted by $\mathsf{L}_{\mathcal{H}}$ because it forms a lattice, a partially ordered set for which the operations are defined between each pair of elements. The partial ordering is given by subspace inclusion. Alternatively (and more naturally), the algebra of quantum-mechanical events can be considered to be a *partial Boolean algebra*, as we will see in section 4. In this case the lattice-theoretic operations are defined only between those elements which are said to be *commeasurable.*

$|\psi\rangle$ is a normed vector in $\mathcal{H}$. It generates a probability measure, p^{ψ}, over the sample space $\mathcal{H}$ through the familiar rule (in Dirac notation):

$$p^{\psi}(\varphi) = |\langle\psi|\varphi\rangle|^2$$

Note that a vector can be represented as the (one-dimensional) subspace that it spans. A subspace, in its turn, can be represented by the projection operator which projects on it. So we will represent a projection operator and its image with the same symbol.

Now, the probability measure p^{ψ} defined via $|\psi\rangle$ satisfies Kolmogorov's axiom 3 only when the subspaces representing the events are orthogonal.
So we have:

(1) $p(\mathbf{0}) = 0$
(2) $p(P^{\perp}) = 1 - p(P)$
(3) $p(P \vee P') = p(P) + p(P')$, when $P \perp P'$,

where $\mathbf{0}$ is the zero subspace.

One remark: given $|\psi\rangle$ we can define a *density operator* (i.e. a bounded, positive operator on the Hilbert space whose trace is 1) with $\rho = |\psi\rangle\langle\psi|$.

In this way we can express the probability measure by

$$p^{\psi}(P) = \langle\psi|P|\psi\rangle = \sum_i \langle\psi|P|\varphi_i\rangle\langle\varphi_i|\psi\rangle = \sum_i \langle\varphi_i|\psi\rangle\langle\psi|P|\varphi_i\rangle$$
$$\equiv Tr\,[|\psi\rangle\langle\psi|P] = Tr[\rho P],$$

where $\{|\varphi_i\rangle\}$ is any orthonormal basis for $\mathcal{H}$, and in the second equality we used its completeness.

Indeed in this way we gain in generality, because with the density matrix formalism we can represent also mixed states. Moreover **Gleason's theorem** establishes a one-to-one correspondence between density matrices and probability measures which satisfies rules (1), (2) and (3) above. This important theorem makes sure that for any probability measure p there exists a density operator ρ, so that $p(P) = p^{\rho}(P) = Tr[\rho P]$. It also illustrates which kind of probability structure a Hilbert space naturally carries, compare to a simple set.

So we are led to alter the definition of a quantum probability theory, so that it is given by an ordered triple $(\mathcal{H}, \mathsf{L}_{\mathcal{H}}, \rho)$.

Now that we are able to express probability measures over linear subspaces and the corresponding projector operators, note that it's very easy to extend what we learned to observables. It suffices to write an operator, say A, in its *spectral decomposition* form:

$$A = a_1 P^A_{a_1} + a_2 P^A_{a_2} + \ldots = \sum_i a_i P^A_{a_i},$$

where a_i are its eigenvalues and $P^A_{a_i}$ are the projections on the corresponding eigenspace.

We conclude this paragraph quoting Marlow's summarizing statement

> ≪*Quantum theory is simply the replacement in standard probability theory of event-as-subset-of-a-set (abelian, distributive) by event-as-subspace-of-a-Hilbert-space (non-abelian, non-distributive).*≫
>
> A.R.Marlow in *"Orthomodular Structures and Physical Theory"*, Academic Press 1977

2.3 From quantum probability to quantum mechanics

Quantum probability is a consistent mathematical theory, but as yet has no physical content. First of all we need know how to *interpret* the different mathematical objects. But we also want a *time evolution* for our system. In order to extend *quantum probability* to *quantum mechanics* we have to introduce two further postulates:

1. We **identificate** density operators with the state of the system, and operators with physical quantities, so that the eigenvalues are the possible values.

2. We postulate, that the **dynamics** of the (conservative) state is given by the unitary operator $U(t) = e^{-iH/\hbar t}$ **(⇔ Schrödinger's equation)**.

2.4 Interpretations of quantum mechanics

As famously pointed out by Schrödinger, there is a problem arising from quantum mechanics after it 'receives a dynamics': it sometimes happen that QM assigns the 'wrong' state to some systems. This interpretational difficulty of quantum mechanics is referred to as the 'measurement problem'. On this subject you will certainly recall the surviving problems of a notorious cat...

The 'measurement problem'

We want to consider the measurements on a system. Let's say the system is in the state $|a_i\rangle$ and before the measurement process the apparatus is in the state $|M_0\rangle$, indicating that a measurement has not yet been carried out. After the measurement the apparatus is in the state $|M_i\rangle$, indicating that the observed system is in the state $|a_i\rangle$. What if the system being measured is a superimposed state, like for example $c_1|a_1\rangle + c_2|a_2\rangle$? Then the apparatus should end up in a superposition of measurement results!

Initial state	*measurement*	***Final state***
$\|a_i\rangle\|M_0\rangle$	$\longrightarrow$	$\|a_i\rangle\|M_i\rangle$
$(c_1\|a_1\rangle + c_2\|a_2\rangle)\,\|M_0\rangle$	$\longrightarrow$	$c_1\|a_1\rangle\|M_1\rangle + c_2\|a_2\rangle\|M_2\rangle$
(by linearity)		

We cannot assign to the finale state any well-defined measurement result (see Schrödinger's cat: is it dead or alive?).
How must we interpret the **superposition of the measurement results**?
Any interpretation of quantum mechanics has to face the difficulties raised by the 'measurement problem'. Let's see how the 'orthodox' view goes along with them.

2.5 The 'orthodox' view of quantum mechanics

Until this point we considered the core which is common to almost any quantum-mechanical theory. The 'orthodox' interpretation of quantum mechanics is obtained making two further assumption:

- QM is a **complete theory**: The probability measure p^ρ completely characterizes the quantum system. In particular, it is everything we can know about it (no Hidden Variables).
  Mathematically this can be expressed with the **'eigenstate-eigenvalue link'**:

  > **'Eigenstate-eigenvalue link'**: a system in the state ρ has the value a for the observable A iff $p^\rho(P_a^A) = 1$.

- It solves the 'measurement problem' thank to the **projection postulate**:

  > **Projection postulate:** upon measurement of an observable A on a system $\mathcal{S}$ in the state ρ, the state $\mathcal{S}$ 'collapses' to $P_a^A \rho P_a^A / Tr[\rho P_a^A]$ for some eigenvalue a of A. The probability for this collapse is $Tr[\rho P_a^A]$.

A remark we can do to this approach is that the introduction of the projection postulate is somehow artificial. It seems to be an *ad hoc* solution to the 'measurement problem'. The fact is that this postulate cannot be derived from the previous formalism; it's not even on the same level. The projection postulate speaks of 'measurement', a complex concept which is not properly defined, and including such a concept as a primitive notion in the fundamental theory of quantum mechanics is not really satisfactory.

Another aspect of the 'orthodox' view which someone could find unsatisfactory, is that it generates an indeterministic theory. Let's see what is meant by this.

Indeterminism

There are 2 types of indeterminism in the 'orthodox' view, each one is in direct relation with one of the previous assumptions.

- The first is a so-called **'indeterminism of the moment'**, which derives from the fact that we accept the **'eigenstate-eigenvalue link'**, assuming that we are only able to know probability measures. This kind of indeterminism is also called **'indefiniteness'**, suggesting that we don't attribute values to the observables A until the condition $p^\rho(P_a^A) = 1$ for some a is satisfied.

In Hidden Variables theories this will be released, accepting a kind of realism which will lead us to assume **value definiteness** (VD) for all observables anytime.

- The second kind of indeterminism is a **'dynamical indeterminism'** and is directly related with the **projection postulate**, which indeed projects our system in one of the possible states, but it's not possible to tell which one in advance.

All formal and 'philosophical' difficulties arising from the 'orthodox' interpretation can be avoided in different ways, each one generating a different kind of interpretation. Most of the time this is dome releasing the 'eigenstate-eigenvalue link' and proposing an alternative solution to the measurement problem, a part from the projection postulate.

Introducing Hidden Variables is a possible and indeed quite natural alternative to the 'orthodox' view, which in fact would truly simplify the interpretation of quantum mechanics, avoid intrinsic indeterminism, and explain the probabilistic nature of the quantum world in terms of a statistical average over unknown parameters. Of course nobody really believes that this can be obtained for free. . .

3 Hidden Variables theories and no-go theorems

As seen in the previous section, the orthodox view of Quantum Mechanics states QM to be a complete theory. An attack towards this claim came 1935 from Einstein, Podolsky and Rosen in their famous paper "*Can Quantum-Mechanical description of physical reality be considered complete?*" [4].
This motivated the search for alternative interpretations of Quantum Theory, based on the assumption of **Value Definiteness**, which are known as Hidden Variables theories. Such theories, given some (reasonable) assumptions, were proved false: we present in this section the Bell Inequalities and Kochen-Specker Theorem, which prove that Hidden Variables theories cannot be local (Bell) and noncontextual (Specker).

3.1 EPR Paradox

We present and briefly discuss the argumentation of Einstein, Podolsky and Rosen against the completeness of Quantum Mechanics.
In the famous article [4] they first made the following requirement for completeness: "*every element of the physical reality must have a counterpart in the physical theory*". They then gave a criterion to understand what to call physical reality: "*if, without in any way disturbing the system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity*".

But in Quantum Mechanics observables corresponding to non-commuting operators cannot be simultaneously measured: from that point Einstein, Podolsky and Rosen argue that *either* Quantum Mechanics is not complete *or* two physical quantities associated to non-commuting operators cannot have simultaneous reality.

They then study a two particle system to prove that the latter possibility is false (with the criterion for reality given above): let us suppose we have two systems which can interact up to time $t = T$; no more interaction is allowed after that time. We then study the second system. We write the wave function Ψ with respect to the eigenbasis ϕ_n and θ_k of operators A and B, respectively:

$$\Psi(x_1, x_2) = \sum_n a_n(x_1)\phi_n(x_2) = \sum_k b_k(x_1)\theta_k(x_2)$$

At that point the a_n and the b_k are to be considered merely as the coefficients of the expansions into series of orthogonal functions ϕ_n and θ_k.

Suppose now that the quantity A (we use, as usually, the same symbol for the observable and the corresponding operator) is measured: the system undergoes what is known as collapse of the wave function and is left in a pure state $\psi_i(x_2)$. From that we know that now the wave function of the first system is $a_i(x_1)$.

Now Einstein, Podolsky and Rosen show that the a_n and the b_k can be eigenfunctions of non-commuting operators C and D on the first system, so that measuring A or B in the second system allows precise prediction of the value of C respectively D in the first. But the second system has no more interactions with the first, so that a measuring process on it is not disturbing the first system. With the criterion for a physical quantity to be considered real given above, we are now obliged to accept that the two non-commuting observables C and D have simultaneous reality: then QM is not a complete theory.
The problem about this argumentation stays in its definition of reality, where it is assumed that a real quantity must have a definite value at all time: this is what is known as **Value Definiteness** (VD), an assumption that, as we will see, leads to contradiction with Quantum Mechanics.

3.2 Hidden Variables

The quantum formalism contains features which may be considered objectionable by some (e.g. J.S. Bell refers to QM as being "unprofessional in its lack of clarity"): this features are subjectivity and indeterminism.

The aim of a Hidden Variable theory is to give a formalism that, while being empirically equivalent to Quantum Mechanics, does not contain such features. The main assumptions that undergo such theories are **Value Definiteness** (VD) and **Noncontextuality** (NC):

> (VD): All observables defined for a QM system have definite values at all time.
> (NC): If a QM system possesses a property (value of an observable), then it does so independently of any measurement context.

VD is not compatible with the orthodox view: to eliminate indeterminism (which is intrinsic in the 'orthodox' interpretation, naturally emerging in quantum phenomena) it is necessary to introduce some parameters λ (called hidden parameters) that completes the information given from the wave function Ψ, so that the knowledge of a state $|\Psi, \lambda\rangle$ (consisting in the knowledge of both, wave function Ψ and hidden parameter λ), allows one to make precise predictions about individual measuring of on observable O.
The statistical information given by the expectation value $E(O) = \langle\Psi|O|\Psi\rangle$ would be substituted by a precise one-to-one mapping between state and observable value $V_\lambda^\Psi(O)$.

$$|\Psi\rangle \longrightarrow |\Psi, \lambda\rangle \text{ (λ hidden parameter)}$$

$$E(O) = \langle\Psi|O|\Psi\rangle \longrightarrow V_\lambda^\Psi(O) \text{ (definite value)}$$

We want to point out again that such a formalism should be empirically equivalent to Quantum Mechanics, so that the distribution of the hidden parameters $f_\Psi(\lambda)$

for a given Ψ should be such that the mean value $\int f_\Psi(\lambda) V_\lambda^\Psi d\lambda$ would be equal the expectation value $E(O)$; and that $V_\lambda^\Psi(O)$ should have the mathematical features of an expectation value function (being the expectation value for a complete state $|\Psi, \lambda\rangle$).

3.3 Three theorems against the Hidden Variables hypothesis

We present here three theorems against the possibility of HV theories to reproduce the predictions of Quantum Mechanics: John Von Neumann's Theorem, the Bell Inequalities (BI) and the Kochen-Specker Theorem (KS).

John Von Neumann's theorem was believed for a long time to prove the inadequacy of HV theories, while in reality it makes an injustificated assumption and does not reach such a result; BI show that the prediction of Quantum Mechanics and local Hidden Variables theories are different for two spin-1/2 particles in singlet state, and experiments done by Aspect showed results according to QM prediction (and thus against HV predictions); KS shows that QM prediction are not compatible with both VD and NC assumptions for a spin-1 particle.

3.3.1 Von Neumann's Theorem

Von Neumann tried to show that an expectation value function $E(O)$ cannot be of the form $V_\lambda^\Psi(O)$ with a one to one mapping between state and observable measured value. To do so he made some assumptions on the general form of any $E(O)$ and showed that these cannot be fulfilled from a definite value function.

The assumptions regarding the function are as follows.
First, the value E assigns to the identity observable $\mathbb{1}$ is equal to unity:

$$E(\mathbb{1}) = 1 \tag{1}$$

The second assumption is that E of any real linear combination of observables is the same linear combination of the values E assigns to each individual observable:

$$E(aA + bB + ...) = aE(A) + bE(B) + ... \tag{2}$$

Finally, it is assumed that E of any projection operator P must be non-negative:

$$E(P) \geq 0 \tag{3}$$

From these assumptions it follows that the expectation value function must be of the form $E(O) = tr(\rho O)$, where ρ is a positive operator with $tr(\rho) = 1$.
Note that a value map function $V_\lambda^\Psi(O)$ must obey the relation:

$$f(V_\lambda^\Psi(O)) = V_\lambda^\Psi(f(O))$$

where f is any mathematical function. This is easily seen by noting that the quantity $f(O)$ can be measured by measuring O and evaluating f of the result. This means that the value of the observable $f(O)$ is f of the value of (O). Thus, if $V_\lambda^\Psi(O)$ maps each observable to a precise value we must have the relation given above.

Consider now the projection operator P_ϕ projecting onto the vector ϕ: for P_ϕ we have $E(P_\phi) = tr(\rho P_\phi) = \langle\phi|\rho|\phi\rangle$. We also know $P_\phi = P_\phi^2$ (P_ϕ being a projection).
Then the relation given above implies:

$$V_\lambda^\Psi(P_\phi) = V_\lambda^\Psi(P_\phi^2) = V_\lambda^\Psi(P_\phi)^2$$

It then follows that $V_\lambda^\Psi(P_\phi)$ must be equal 0 or 1.

If $E(P_\phi)$ takes the form of a value map function $V_\lambda^\Psi(P_\phi)$, then it follows that the quantity $\langle\phi|\rho|\phi\rangle$ is equal either 0 or 1. Varying continuously $|\phi\rangle$ also $\langle\phi|\rho|\phi\rangle$ will change continuously. It follows then that the quantity $\langle\phi|\rho|\phi\rangle$ must be constant equal 0 or 1.
In the first case we would have $\rho = 0$ and $tr(\rho\mathbb{1}) = 0$ in contradiction to (1), in the second $\rho = \mathbb{1}$ and $tr(\rho\mathbb{1}) = n$, where n is the dimensionality of the space, again in contradiction with (1).

Does this represent a definitive proof against HV?
No, because condition (2) is too strong: there is no reason why such an assumption should be made for non-commuting operators. For instance in the case of the measurement of the spin of a spin-1/2 particle in a direction lying in the xy-plane at 45° from each axis $\sigma' = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)$. The eigenvalues of this observables are $\pm\frac{1}{2}$ and therefore, because of $\pm\frac{1}{2} \neq \frac{1}{\sqrt{2}}(\pm\frac{1}{2} \pm \frac{1}{2})$, assumption (2) cannot hold.

We will see that Kochen and Specker make assumption (2) only for sets of commuting observables.

3.3.2 The Bell Inequalities

To formulate Bell's Inequalities we consider two spin-1/2 particles flying in opposite directions. Two physicists, Alice and Bob, can measure the spin components of the first, respectively the second particle in three linear independent directions (not necessary orthogonal) a, b and c. If Alice measures the first particle to have spin $+\frac{1}{2}$ in direction a, then, due to the correlation between the two particles, Bob will measure the spin component in direction a of the second particle to be $-\frac{1}{2}$.

In a HV theory (where VD holds) the result of every measurement must be predictable. In our case the knowledge of the state $|\Psi, \lambda\rangle$ of the particle should allow us to predict the result of a measurement in each direction a, b or c, i.e. every particle should have a given + or − spin component in each of the three directions (this is like saying that, given Ψ and λ, we know in which octant lies the spin of the particle).

There are then eight possible spin combinations, each of which will happen with a given probability P_i.

Probability	Particle A	Particle B
P_1	+a,+b,+c	-a , -b , -c
P_2	+a,+b, -c	-a , -b ,+c
P_3	+a, -b ,+c	-a ,+b, -c
P_4	+a, -b , -c	-a ,+b,+c
P_5	-a ,+b,+c	+a, -b , -c
P_6	-a ,+b, -c	+a, -b ,+c
P_7	-a , -b ,+c	+a,+b, -c
P_8	-a , -b , -c	+a,+b,+c
$\Sigma P_i = 1$		

For example P_1 is the probability that the spin of the first particle lies in the up-front-right octant and that of the second particle in the down-back-left octant.

Alice and Bob are allowed to make only one measurement per particle: we note $P(xd_i, yd_j)$ the probability for Alice to find the result x measuring in d_i direction and for Bob to find result y measuring in d_j direction. Then $P(+a, +b)$ is the probability that the first particle has spin component $+\frac{1}{2}$ in a direction and the second particle has spin component $+\frac{1}{2}$ in b direction. We find:

$$P(+a, +b) = P_3 + P_4 \leq (P_3 + P_7) + (P_2 + P_4)$$

$$P(+c, +b) = P_3 + P_7$$

$$P(+a, +c) = P_2 + P_4$$

In an HV theory must then hold the Bell Inequality:

$$P(+a, +b) \leq P(+a, +c) + P(+c, +b)$$

For $\theta_{(a,b)} = 120°, \theta_{(a,c)} = \theta_{(b,c)} = 60°$ QM calculations do not satisfy the inequality with $\frac{3}{4} \leq \frac{1}{2}$.
Experiments done by A. Aspect show the Bell Inequalities to be violated. This fact excludes the validity of local HV theories.

3.3.3 The Kochen Specker Theorem

The theorem of Kochen Specker states a contradiction between VD and NC for every QM system, and provides a concrete example showing that it is impossible to coherently assign an a priori value to the square of the spin component of a spin-1 particle in 117 direction in space. VD (under the further assumption of NC) leads to contradiction with Quantum Mechanics.

The proof was more recently simplified, and the number of observable reduced to 33. The Kochen-Specker theorem is stronger than Bell's one, because the given example is a one particle experiment. This means that the argument does not involve locality assumptions.
We give here the statement of the theorem:

Let $\mathcal{H}$ be a Hilbert Space of QM state vectors of dimension $x \geq 3$. Let M be a set containing y observables, defined by operators on $\mathcal{H}$. Then, for specific values of x and y, the following two assumptions are contradictory:

> (KS1): All y members of M simultaneously have values, i.e. are unambiguously mapped onto real unique numbers (designated, for observables $A, B, C, \ldots$ by $V_\lambda^\Psi(A), V_\lambda^\Psi(B), V_\lambda^\Psi(C), \ldots$).
> (follows from VD)
>
> (KS2): Values of observables conform to the following constraints:
>
> - If A, B, C are all compatible and $C = A + B$, then $V_\lambda^\Psi(C) = V_\lambda^\Psi(A) + V_\lambda^\Psi(B)$;
> - If A, B, C are all compatible and $C = A.B$, then $V_\lambda^\Psi(C) = V_\lambda^\Psi(A).V_\lambda^\Psi(B)$.
>
> (follows from VD & NC)

The assumption KS1 is a translation of VD, while KS2 follows from NC and again VD.

The following sections contain further discussion about KS theorem: in section 4 the logical-mathematical structure of Quantum Mechanics is investigated, and this will represent the conceptual background that undergoes a deep understanding of the theorem. In section 5 a proof of the theorem involving a spin-1 particle will be presented, and the consequence of KS Theorem are treated in section 6.

4 The quantum logic interpretation

After considering Hidden Variable theories in general and stating the major no-go theorems let's consider quantum logic. Quantum logic furnishes a language which will allow us to formulate an alternative interpretation of quantum mechanics, but also to better express and understand the Kochen-Specker theorem in terms of 'non-reducibility' of quantum mathematical structures to the corresponding classical ones. Physically this 'non-reducibility' will coincide with the impossibility of endowing quantum mechanics with a certain class of hidden variables. The pioneers who first explored this logical structures arising from quantum mechanics are Kochen and Specker [7]

4.1 From quantum probability to quantum logic

In paragraph 2.2 on quantum probability we have already hinted at the basic idea of Quantum logic, which consists in assuming that the lattice-theoretic operations of *meet* ($\wedge$), *join* ($\vee$), and *orthocomplement* ($^{\perp}$) are the 'true' logical operations. For example, to asses the truth of the statement *'P or Q'*, we must represent it as *'$P \vee Q$'*, and so on.

This kind of logic has different features than the classical one. As it can be deduced from the axiom of quantum probability which corresponds to Kolmogorov's axiom 3, we can compare only **orthogonal (compatible) statements**, which correspond to commuting projections (denoted by PcP', which means $[P, P'] = 0$). We don't have such a feature in classical logic, but we have to admit that incompatibility between propositions may also be said to occur in natural languages, like Specker [10] noted. Difficulties arising from propositions of the type "If two times two are five, then there exist centaurs" seem to be due as much to incompatibility as to material implication.

If we think of the quantum world another example of incompatible propositions could be "The particle is in x" and "The particle has momentum p", which clearly cannot be logically connected because of Heisenberg's uncertainty principle. Quantum logic thus automatically tells us which propositions can be logically related, and this is done by the operator c, which verifies their commeasurability.

The mathematical structures which naturally describe all this are **Partial Boolean Algebras**, which we will examine in paragraph 4.3, after considering some features of the Quantum logic interpretation.

4.2 Quantum logic and the 'measurement problem'

Quantum logic interpretation somehow **denies indefiniteness**, because according to it every observable has a well-defined value: according to quantum logic saying 'A has some definite value' is equivalent to 'A has value a_1 or A has value a_2, ...'. Expressed in mathematical terms:

$$P^A_{a_1} \vee P^A_{a_2} \vee \ldots = \bigvee_i P^A_{a_i} = \mathbb{1} \quad \text{by completeness.}$$

Thus the statement is true, because $p^\rho(\mathbb{1}) = 1$, for any ρ.

We have to pay attention to the fact that denying indefiniteness in the quantum logical sense doesn't mean accepting Value Definiteness (VD), which in fact is an attempt to reduce quantum mechanics to classical mechanics.

Analogously quantum logic **solves the 'measurement problem'**. In fact the statement 'for some i, the apparatus is in state $|M_i\rangle$' is true in this interpretation, in the same way the previous statement 'A has some definite value' was true.

From this point of view the 'measurement problem' seems only to be due to the fact that we refer to the quantum world in classical terms. If we indeed assume that the quantum world works with quantum logic the difficulty of indefiniteness disappears, and with it the difficulty in interpreting measurement results. To be sincere the problem has not yet completely disappeared, but we expressed it in terms of explaining the arising of classical phenomena from quantum phenomena. We could suppose that we obtain classical logic from quantum logic in the same way we obtain classical physics from quantum physics going to high quantum numbers.

A stronger position could be the statement that in fact physical processes determine the logical structures which are supposed to describe these processes. It should then be clear that to 'different physics there correspond different logics'. This point of view is actually inspired from similar considerations proposed by Deutsch et al. [2] about the nature of computation.

We don't want to go further down this way, because we have to admit that we already lingered quite long in the misty lands between physics and philosophy. So let's go back to earth and briefly take a look at the mathematical structures which naturally support quantum logic: Partial Boolean algebras.

4.3 Partial Boolean algebras

The concept of partial Boolean algebra (pBa) is a generalization of the concept of Boolean algebra (or total Boolean algebra, tBa), taking into account the fact that not all elements of the algebra can be logically related. Let's consider the following scheme from a Proseminar of Prof Specker himself:

General definition:

A **partial Boolean algebra** is a set L containing at least two different elements 0 and 1, supplied with four operations $c, \neg, \wedge, \vee$, such that the following axioms hold:

1. c is reflexiv (XcX) and symmetric
2. If XcY, so $X \vee Y$, $X \wedge Y$ are defined.
   $\neg$ is total
3. If XcY, XcZ, YcZ, so the operations $0, 1, \neg, \wedge, \vee$ generate a Boolean lattice (i.e. $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$, and so on.)

Let's see an example that we already know from our paragraph on quantum probability theory:

Example: Linear subspaces of $\mathbb{R}^n$

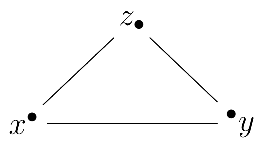
For $X, Y \subset \mathbb{R}^n$ linear subspaces, we define:

- $X \wedge Y := X \cap Y$
- $X \vee Y := span\{X \cup Y\}$
- We say XcY, if $X = Z \vee A$ and $Y = Z \vee B$ for $A \perp B$
- Whole space: Element 1.
- $\{0\}$: Element 0.
- $\neg X := X^{\perp}$ (total complement)

If we now consider $\mathbb{R}^3$, we have four types of linear subspaces: $\{\mathbf{0}\}$ which corresponds to 0, one-dimensional subspaces , two-dimensional subspaces, and $\mathbb{R}^3$ itself which corresponds to 1.

We want now to graphically represent the relation between three mutually orthogonal one-dimensional subspaces, spanned by the vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ respectively. Each vertex in the Figure below corresponds to such a subspace. That

two subspaces are orthogonal means in these case that their projections are commeasurable, which is represented connecting the corresponding vertices.
We represent with $u(x)$ the one-dimensional subspace spanned by $\vec{x}$ and with $d(x)$ the two-dimensional subspace which is orthogonal to it, and we do the same for y and z.



$$\begin{aligned} \text{1-dim}: \ & u(x) = span\{\vec{x}\}, \\ & \dots \\ \text{2-dim}: \ & d(x) = span\{\vec{x}\}^{\perp} = u(y) \vee u(z), \\ & \dots \end{aligned}$$

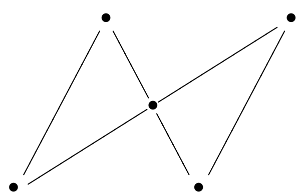
It'easy to check that the set $\mathsf{L} = \{0,\ 1,\ u(i),\ d(i) \ \mid \ i = x, y, z\}$ with the operations $\neg, \wedge, \vee$ forms a total Boolean algebra. In fact all the elements of the partial Boolean algebra are connected.
We could for example check distributivity like this:

$$d(x) \wedge \underbrace{\big(u(x) \vee u(y)\big)}_{=d(z)} = d(x) \wedge d(z) = u(y)$$

$$\underbrace{\big(d(x) \wedge u(x)\big)}_{=0} \vee \underbrace{\big(z(x) \wedge u(y)\big)}_{=u(y)} = u(y).$$

The fact that the pBa contains $2 + 2 \cdot 3 = 2^3$ elements confirms that it is a total Boolean algebra (any total Boolean algebra has 2^n elements).

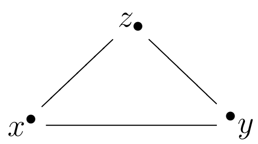
Let us now consider the pBa of subspaces of $\mathbb{R}^3$ represented by the following diagram:



This partial Boolean algebra contains $2 + 2 \cdot 5 = 12$ elements. This means that it cannot be a total Boolean algebra. But we could try to **embed** it in a tBa thanks to a homomorphism that maps the pBa to a tBa, preserving the logical connections of the elements of the pBa.

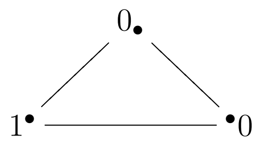
In addition we know from Boole's theory that every total Boolean algebra has an homomorphism to the set $\{0, 1\}$. This means that every pBa with an homomorphism to a tBa can be embedded to $\{0, 1\}$. Let's see how such an embedding should look for the first diagram we saw.

Example: homomorphism $h : pBa \longrightarrow \{0, 1\}$



We know:

$$u(x) \vee u(y) \vee u(z) = 1$$
$$u(x) \wedge u(y) = u(x) \wedge u(z) = u(y) \wedge u(z) = 0.$$



This means that our homomorphism should be:

$h\big(u(i)\big) =: f(i)$ equal '1' for exactly one point of the triangle.

On the other hand, to such a **coloration** f which assigns '1' to a unique point of the triangle, we can define an homomorphism h in this way:

$$h(u(i)) = f(i), \qquad h(d(i)) = \neg f(i).$$

Thus we reduced the embedding of a pBa to a coloration problem.

The **Kochen-Specker theorem** now asserts that not any partial Boolean algebra can be embedded in a total Boolean algebra. This is equivalent to the assertion that not any partial Boolean algebra can be colored so that only one point of a triangle is assigned the value '1', while the other two are assigned the value '0'.

We can express this in a deeper way, saying that quantum logic is not 'reducible' to classical logic, for example assigning it Hidden Variables. Let's try to understand better this last assertion explaining the relation between a coloration and an assignment of Hidden Variables, and examining how to formulate the Kochen-Specker theorem in this context.

5 Proof of the Kochen-Specker Theorem

The proof of the Kochen-Specker Theorem now works in two steps: to see that the statements KS1 and KS2 given in 3.3.3 leads to a coloring problem, and to prove that this coloring problem cannot be solved.

The original Kochen-Specker proof operates in a 3 dimensional Hilbert space $\mathcal{H}^3$. We consider an arbitrary operator O with three distinct eigenvalues a_1, a_2 and a_3, its eigenvectors $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$, and projection operators P_{ϕ_1}, P_{ϕ_2} and P_{ϕ_3} projecting on the rays spanned by these vectors.
Now, P_{ϕ_1}, P_{ϕ_2} and P_{ϕ_3} are mutually commuting operators and we can then apply KS2:

$$V_\lambda^\Psi(P_{\phi_i}) = V_\lambda^\Psi(P_{\phi_i}^2) = V_\lambda^\Psi(P_{\phi_i})^2 \qquad (KS2)$$

It then follows $V_\lambda^\Psi(P_{\phi_i}) = 0$ or 1. From $\sum_i P_{\phi_i} = \mathbb{1}$ and KS2 it follows that for exactly one of the $|\phi_i\rangle$, $V_\lambda^\Psi(P_{\phi_i}) = 1$ and for the other two $V_\lambda^\Psi(P_{\phi_i}) = 0$. The arbitrary choice of O selects in an arbitrary way the three $|\psi_i\rangle$, which in turn select rays in $\mathcal{H}^3$. A different choice of O would lead to different rays. So VD, and in particular KS1, have as a consequence that for a given triple of orthogonal rays in $\mathcal{H}^3$, to exactly one of them must be assigned value 1, to the others 0.

The reader should recognize the logical structure: the problem has now exactly the form given in paragraph 4.3, if such a coloring does exist, the pBa generated by the events on $\mathcal{H}^3$ can be embedded in a total one.

It is possible to show that if the presented coloring problem can be solved for $\mathcal{H}^3$, then it can be solved for $\mathbb{R}^3$. Counterpositively, if it is impossible to find such a coloring for $\mathbb{R}^3$, then it is impossible also for $\mathcal{H}^3$. Our problem has been reduced to the problem of coloring every point of the two-sphere, such that for triples of orthogonal points, exactly to one of them is assigned value 1.

It should be stressed that at this point there is no connection between $\mathbb{R}^3$ and the physical space: KS wish to show that for any QM system requiring a representation of dimension ≥ 3, the ascription of values in conjunction with KS2 leads to a contradiction, and in order to do so it is sufficient to consider the space $\mathbb{R}^3$.

KS proceeded this way, but they illustrate with an example that does establish a relationship with physical space. Consider a spin-1 particle. While the operators corresponding to the square of the spin components do not commute, their square do. Further we have:

$$S_x^2 + S_y^2 + S_z^2 = 2\mathbb{1}$$

so that condition KS2 implies two of them to be 0, and the other to be 1. The choice of the reference system chooses different directions for the x, y ,z axis, so that this condition must be true for every triple of orthogonal directions in $\mathbb{R}^3$. It is evident that this coloring problem is equivalent to the preceding one.

We now proceed with the proof that such a coloring is impossible: for simplicity let us say that assigning a point value 0 is like to color it white and to assign it value 1 is like to color it black. For an orthogonal triple exactly one point should then be colored white.

The first step is to see that close points must have the same color. Consider the diagram shown in Figure 1: vectors are represented by dots, and orthogonal vectors are connected by lines.

It is an easy geometrical problem to show that such a diagram is realizable in

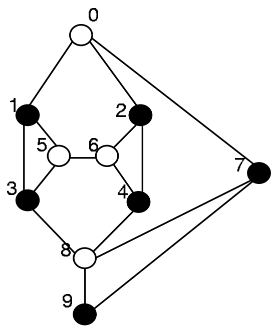


Figure 1: Ten-point KS graph

space if the vectors 0 and 9 are separated from an angle $\leq arcsin(1/3) \approx 19.5°$. Imagine (for *reductio ad absurdum*) to have two vectors separeted by an angle $\leq arcsin(1/3)$ and colored with different colors: construct a diagram such that the white vector is 0 and the black one is 9: we are then forced to color the diagram as showed in Figure 1, and to have 5 and 6 (which are connected and thus orthogonal) both colored white, which is a contradiction. Hence, two vectors closer than $arcsin(1/3)$ cannot have different colors.

Now let us construct the diagram of Figure 2: P, Q and R are three orthogonal rays; place between each two of them five realizations of the diagram of Figure 1 with an angle of 18° identifying the 0 point of one diagram with the 9 point of the preceding one. That the diagram is constructable is directly born out by the construction itself.

The diagram is constructable but not consistenly colorable: from our first step we know that point 0 of a diagram must be same color as point 9, that is identified with point 0 of the neighboring diagram. By repetition of the argument it is clear that all the points P, Q and R must be the same color. But they are orthogonal and thus the diagram is not consistently colorable.

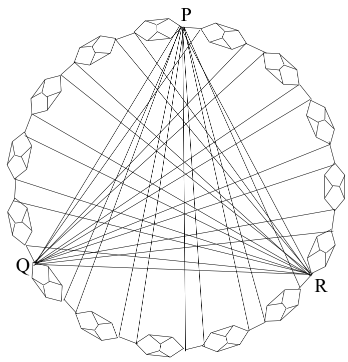


Figure 2: 117-point KS graph

6 Conclusion

Kochen and Specker Theorem proves that every quantum mechanical system of dimension greater or equal than three has an intrinsic indeterminism, i.e. that the logical structure (*quantum* logical structure!) that describes such a system (a partial boolean algebra) cannot be described with help of a classical one (it's impossible to embed this partial boolean algebra into a total one).
The proof of the theorem acts on a particular system such that orthogonality in real physical space corresponds to orthogonality in quantum mechanical sense, this way the intuition is helped and the understanding is much easier.
It should however be stressed that the theorem does not direct only to this particular system, but to *every quantum mechanical system of dimension greater or equal than three*. Every such system cannot be described in terms of noncontextual Hidden Variables theories.
In Simon Kochen's and Ernst Specker's words: "If a physicist X believes in hidden variables...... the prediction of X contradicts the prediction of Quantum Mechanics".

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