An H-theorem for the general relativistic Ornstein-Uhlenbeck process

M. Rigotti[a] and F. Debbasch[b]
LERMA (ERGA), UMR 8112, UPMC, Site Le Raphaël, 3 rue Galilée 94200 Ivry sur Seine, France

(Received 9 June 2005; accepted 26 July 2005; published online 28 October 2005)

We construct conditional entropy four-currents for the general relativistic Ornstein-Uhlenbeck process and we prove that the four-divergences of these currents are always non-negative. This H-theorem is then discussed in detail. In particular, the theorem is valid in any Lorentzian space-time, even those presenting well-known chronological violations.
[DOI: 10.1063/1.2038627]

NOTATIONS

In this article, c denotes the speed of light, and the signature of the space-time metric is $(+,-,-,-)$. Indices running from 0 to 3 are indicated by Greek letters. Latin letter indices run instead from 1 to 3. We also introduce the abbreviation $\partial_p^\mu = \partial / \partial p_\mu$ for the partial derivative with respect to an arbitrary component of the momentum p. This notation underlines the fact that this operator transforms as a contravariant vector. Similarly we will often write $\partial_\mu = \partial / \partial x^\mu$, but the latter operator naturally does *not* transform as a tensor. Finally, det g stands for the determinant of the coordinate basis components of the metric tensor g.

I. INTRODUCTION

In Galilean physics, the most common way to quantify the irreversibility of a phenomenon is to introduce an entropy, i.e., a functional of the time-dependent thermodynamical state of the system which never decreases with time. In usual Galilean continuous media theories, the total entropy $\mathcal{S}$ can be written as the integral of an entropy density s over the volume occupied by the system.[24] One also introduces an entropy current $\mathbf{j}_s$ and, since entropy is by definition not generally conserved, the relation $\partial_t s + \nabla \cdot \mathbf{j}_s \geqslant 0$ holds for every evolution of the system.

Traditional relativistic hydrodynamics and kinetic theory deal with the problem in a completely similar manner. An entropy four-current S is associated to the local thermodynamical state of the system;[5,21,14] the total entropy $\mathcal{S}(t_0)$ of the system at time-coordinate $t=t_0$ can be obtained by integrating S over the three-dimensional space-like submanifold $t=t_0$ and the entropy fluxes are obtained by integrating S over two-dimensional submanifolds of space-time. Since entropy is not generally conserved, the simple relation $\nabla \cdot S = \nabla_\mu S^\mu \geqslant 0$ holds for any evolution of the system.

Actually, given a system and its dynamics, any four-vector field S of non-negative divergence which depends on the local thermodynamical state of the system can be considered as an entropy current. In particular, nothing precludes the possibility of associating more than one entropy current to a single local state of a system.

Let us illustrate this remark by considering two special cases of great physical and mathematical interest. Historically speaking, the first statistical theory of out-of-equilibrium systems is Boltzmann's model of dilute Galilean gases.[4,24,13] The local state of the system is encoded in the so-called one particle distribution function f, which obeys the traditional Boltzmann equation. A

[a]On leave from ETH-Hönggerberg, CH-8093 Zürich, Switzerland.
[b]Author to whom correspondence should be addressed; electronic mail: fabrice.debbasch@wanadoo.fr

direct consequence of this equation is that a certain functional of the distribution function never decreases with time. Boltzmann denoted this functional by H and the result is therefore known as Boltzmann's H-theorem. To this day, H is the only-known functional of f that never decreases in time. This H-theorem has later on been extended to the relativistic generalization of Boltzmann's model of dilute gases.[14] Thus, the relativistic Boltzmann gas also admits an entropy (and an entropy current) and it seems that this entropy is unique.

The situation is drastically different for stochastic processes. Indeed, a theorem due to Voigt[26,22] states that, under very general conditions, a stochastic process admits an infinity of entropies: Let X be the variable whose time-evolution is governed by the stochastic process and let dX be a measure in X-space $\mathcal{X}$ (typically, dX is the Lebesgue measure if $\mathcal{X}\subset\mathbb{R}^n$). Now let f and g be any two probability distribution functions solutions of the transport equation associated to the stochastic process. Then, the quantity

$$\mathcal{S}_{f|g}(t) = -\int_{\mathcal{X}} f(t,X)\ln\left(\frac{f(t,X)}{g(t,X)}\right)dX \tag{1}$$

is a never decreasing function of time and is called the conditional entropy of f with respect to g. Thus, to any given $f(t,\cdot)$ representing the state of the system at time t, one can associate as many entropies as there are different solutions g of the transport equation so, typically, an infinity. Naturally, if the function g_0 defined by $g_0(t,X)=1$ for all t and X is a solution of the transport equation, the conditional entropy $\mathcal{S}_{f|g_0}$ of any distribution f with respect to g_0 coincides with the Boltzmann entropy of f.

The notion of conditional entropy corresponds to what is sometimes called the Kullback information and we refer the reader to Refs. 19, 18, and 3 for extensive discussions of this concept.

The application of Voigt's theorem to Galilean stochastic processes is of course straightforward and rather well known, but its application to relativistic stochastic processes demands discussion. To be definite, we will now particularize our treatment to the ROUP, which is the first relativistic process to have been introduced in the literature.[7,8,2,1,6]

Given a reference frame (chart) $\mathcal{R}$, the ROUP transcribes as a set of stochastic equations governing the evolution of the position and momentum of a diffusing particle as functions of the time coordinate t in $\mathcal{R}$. This set of equations is a stochastic process in the usual sense of the word, and Voigt's theorem ensures this process admits an infinity of conditional entropies. But, by construction, these entropies *a priori* depend on the reference frame $\mathcal{R}$ and the general theorem does not furnish any information about their tensorial status.

This question has been partly answered for the special relativistic Ornstein-Uhlenbeck process.[1] In flat space-time, the ROUP admits as invariant measure in p-space a Jüttner distribution J;[16] this distribution simply describes a special relativistic equilibrium at the temperature of the fluid surrounding the diffusing particle. It has been shown in Ref. 1 that this Jüttner distribution can be used to construct a four-vector field of non-negative four-divergence which can be interpreted as the conditional entropy current of f with respect to J.

The aim of the present article is to prove the existence of conditional entropy currents for the ROUP in curved space-time. The matter is organized as follows. Section II reviews some basic results pertaining to the ROUP in curved space-time with particular emphasis on the Kolmogorov equation associated to the process. It is also recalled here that, in a generic space-time, this equation does *not* admit any equilibrium stationary solution.[6] In particular, a general relativistic Jüttner distribution is not, generically, a solution of the Kolmogorov equation and, therefore, cannot be used to construct an entropy current in curved space-time. We therefore consider two arbitrary solutions f and g of the Kolmogorov equation and introduce in Sec. III A a candidate for the conditional entropy current of f with respect to g. We then prove in Sec. III B that the four-divergence of this current is always non-negative. This is our main result and it constitutes an H-theorem for the ROUP in curved space-time. Note that the flat space-time version of this H-theorem is itself a new result because our previous work[1] only proved the existence of a single entropy current for the ROUP in flat space-time, i.e., the conditional entropy current of an arbitrary distribution f with respect to the Jüttner equilibrium distribution J. Finally, the new H-theorem and some of its possible extensions are discussed at length in Sec. IV. The Appendix recalls and, if necessary, proves some simple but important purely geometrical relations useful in deriving the H-theorem.

II. BASICS ON THE ROUP IN CURVED SPACE-TIME

A. Kolmogorov equation

The general relativistic Ornstein-Uhlenbeck process can be viewed as a toy model for the diffusion of a point particle of nonvanishing mass m interacting with both a fluid and a gravitational field. This process is best presented by its Kolmogorov equation in manifestly covariant form.[6] The extended phase-space is the eight-dimensional bundle cotangent to the space-time manifold with local coordinates, say (x^μ, p_ν), $(\mu,\nu)\in\{0,1,2,3\}^2$. At each point in space-time, the four-dimensional (4D) momentum space $\mathcal{P}$ is equipped with the 4D volume measure:

$$\mathcal{D}^4 p = \theta(p_0)\,\delta(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}}d^4p, \tag{2}$$

with $d^4p = dp_0 \wedge dp_1 \wedge dp_2 \wedge dp_3$. This measure behaves as a scalar with respect to arbitrary coordinate changes. Note that integrals over $\mathcal{P}$ defined by using (multiples of) $\mathcal{D}^4p$ as a measure are *de facto* restricted to the (generally position-dependent) mass-shell.

Let f be the probability distribution function in the extended phase-space of a particle diffusing in a surrounding fluid with normalized four-velocity U. As shown in Ref. 6, f obeys a manifestly covariant Kolmogorov equation which can be written in the following compact form:

$$\partial_\mu(p^\mu f) = -\partial_p^\mu\left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\}. \tag{3}$$

The coefficients $\widetilde{\Gamma}_\mu$, which do not constitute a tensor, are defined by

$$\widetilde{\Gamma}_\mu = \Gamma^\lambda_{\mu\nu} g^{\kappa\nu} p_\kappa p_\lambda \tag{4}$$

and

$$\mathcal{K}_\mu(f) = I_\mu f - \partial_p^\nu (J_{\mu\nu} f) \tag{5}$$

with

$$I_\mu = -DK^{\alpha\ \ \beta}_{\ \ \mu\ \ \nu}\partial_p^\nu\left(\frac{p_\alpha p_\beta}{p\cdot U}\right) + mcF_\mu, \tag{6}$$

$$J_{\mu\nu} = -DK^{\alpha\ \ \beta}_{\ \ \mu\ \ \nu}\frac{p_\alpha p_\beta}{p\cdot U}. \tag{7}$$

The tensor K is independent of p. It depends on U and on the metric g, but only through the projector Δ on the orthogonal to U, which reads:

$$\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu. \tag{8}$$

The explicit expression of K in terms of U and Δ is

$$K^{\alpha\mu\beta\nu} = U^\alpha U^\beta \Delta^{\mu\nu} + U^\mu U^\nu \Delta^{\alpha\beta} - U^\alpha U^\nu \Delta^{\mu\beta} - U^\mu U^\beta \Delta^{\alpha\nu}. \tag{9}$$

Finally, F represents the deterministic part of the force exerted by the fluid on the diffusing particle; its expression as a function of p and U reads

$$F_{\mu} = -\lambda_{\mu\nu} p^{\nu} \frac{p^2}{m^2 c^2} + \lambda_{\alpha\beta} \frac{p^{\alpha} p^{\beta}}{m^2 c^2} p_{\mu}, \tag{10}$$

with

$$\lambda_{\mu\nu} = \frac{\alpha (mc)^2}{(p \cdot U)^2} \Delta_{\mu\nu}, \tag{11}$$

$\alpha > 0$ being the friction coefficient (see Ref. 7). Note that F is by construction orthogonal to p.

It has been shown in Ref. 6 that (3) does not generically admit stationary solutions. In particular, a general relativistic Jüttner distribution cannot be used to construct in curved space-time a preferred conditional entropy current for the ROUP.

III. *H*-THEOREM FOR THE ROUP IN CURVED SPACE-TIME

A. Definition of the conditional entropy currents

Given any two probability distribution functions f and g defined over the extended phase-space, a natural definition for the conditional entropy current of f with respect to g is

$$S_{f|g}(x) = -\int_{\mathcal{P}} p f(x,p) \ln\left(\frac{f(x,p)}{g(x,p)}\right) \mathcal{D}^4 p. \tag{12}$$

This definition is clearly the simplest generalization of Eq. (37) in Ref. 1 to both an arbitrary reference distribution g and a possibly curved space-time background.

We will now prove that for all f and g solutions of the Kolmogorov equation (3), the four-divergence of $S_{f|g}$ is non-negative.

B. Proof of the *H*-theorem

The proof of the H-theorem for the general relativistic Ornstein-Uhlenbeck process will be carried out in two steps.

1. Computation of the four-divergence of the entropy current

Theorem 1: *For any f and g solutions of Kolmogorov equation*

$$\nabla \cdot S_{f|g}(x) = \int_{\mathcal{P}} J_{\mu\nu}(x,p) D^{\mu}[f/g] D^{\nu}[f/g] \mathcal{D}^4 p, \tag{13}$$

where J is defined by (7) *and the functional D is given by*

$$D^{\mu}[f/g] = \partial_p^{\mu} \ln(f/g). \tag{14}$$

Proof: The main idea behind the proof is to use Kolmogorov equation (3) to convert all the spatial derivatives into derivatives with respect to momentum components. To do this we will deal with various integrals over $\mathcal{P}$ by integrating most of them by parts. This procedure generally leads to the appearance of so-called "border terms." Some of them trivially vanish if we suppose, as is customary in statistical physics, that phase-space distribution functions tend to zero sufficiently rapidly at infinity (in 4D p-space). One is then left with border terms that are to be evaluated on the hyperplane $p \cdot U = 0$. These also vanish for the following reason. Let us choose, at each point in space-time, an orthornormal basis (tetrad) (e_a), $a = 0,1,2,3$ in the tangent space. Introducing the components p_a and U^a of p and U in this base, the normalization condition $U^2 = 1$ reads:

$$U^0 = \sqrt{1 + \sum_{i=1}^{3} (U^i)^2} \tag{15}$$

so that

$$U^0 > \sqrt{\sum_{i=1}^{3} (U^i)^2}. \tag{16}$$

The condition $p \cdot U = 0$ becomes $p_0 U^0 + \Sigma_{i=1}^{3} p_i U^i = 0$; since $U^0 > 0$, this translates into

$$p_0 = -\frac{\sum_{i=1}^{3} p_i U^i}{U^0}. \tag{17}$$

It follows easily from (16) and (17) that $(p_0)^2 < \Sigma_{i=1}^{3} (p_i)^2$ on the hyperplane $p \cdot U = 0$. The Dirac δ distribution which enforces the on mass-shell restriction $p^2 = m^2 c^2$ therefore vanishes on the hyperplane $p \cdot U = 0$, ensuring that the corresponding border terms disappear.

Let us now proceed with the proof of Theorem 1. Direct derivation of Eq. (12) leads to

$$\begin{aligned}\nabla_\kappa S^\kappa_{f|g} &= -\partial_\kappa \int_{\mathcal{P}} p^\kappa f \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p - \Gamma^\alpha_{\alpha\kappa} \int_{\mathcal{P}} p^\kappa f \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p \\ &= \underbrace{-\int_{\mathcal{P}} \partial_\kappa (p^\kappa f) \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p}_{=\mathcal{A}_1} \underbrace{-\int_{\mathcal{P}} p^\kappa \left[(\partial_\kappa f) - \frac{f}{g}(\partial_\kappa g)\right] \mathcal{D}^4 p}_{=\mathcal{A}_2} \\ &\quad \underbrace{-\int_{\mathcal{P}} p^\kappa f \ln\left(\frac{f}{g}\right) \partial_\kappa (\mathcal{D}^4 p)}_{=\mathcal{A}_3} \underbrace{-\Gamma^\alpha_{\alpha\kappa} \int_{\mathcal{P}} p^\kappa f \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p}_{=\mathcal{A}_4}.\end{aligned} \tag{18}$$

Using Kolmogorov equation (3), integrating by parts, and inserting the definition of $\mathcal{K}_\mu(f)$ Eq. (5) we obtain for $\mathcal{A}_1$:

$$\begin{aligned}\mathcal{A}_1 &= \int_{\mathcal{P}} \partial^\mu_p \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p = -\int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \left[I_\mu f - \partial^\nu_p (J_{\mu\nu} f)\right]\right\} \partial^\mu_p \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p \\ &\quad - \int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \ln\left(\frac{f}{g}\right) \partial^\mu_p (\mathcal{D}^4 p).\end{aligned} \tag{19}$$

Let us now consider the term $\mathcal{A}_2$:

$$\begin{aligned}\mathcal{A}_2 &= -\int_{\mathcal{P}} p^\kappa \left[(\partial_\kappa f) - \frac{f}{g}(\partial_\kappa g)\right] \mathcal{D}^4 p \\ &= \underbrace{-\int_{\mathcal{P}} \partial_\kappa (p^\kappa f) \mathcal{D}^4 p}_{=\mathcal{B}_1} \underbrace{+\int_{\mathcal{P}} \partial_\kappa (p^\kappa g) \frac{f}{g} \mathcal{D}^4 p}_{=\mathcal{B}_2}.\end{aligned} \tag{20}$$

Using again Kolmogorov equation (3) and integrating by parts, we obtain for the term $\mathcal{B}_1$:

$$\mathcal{B}_1 = \int_{\mathcal{P}} \partial^\mu_p \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \mathcal{D}^4 p = -\int_{\mathcal{P}} \left\{\widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(f)\right\} \partial^\mu_p (\mathcal{D}^4 p), \tag{21}$$

and for the term $\mathcal{B}_2$:

$$\mathcal{B}_2 = -\int_{\mathcal{P}} \partial_p^\mu \{\widetilde{\Gamma}_\mu g + \mathcal{K}_\mu(g)\} \frac{f}{g} \mathcal{D}^4 p = \int_{\mathcal{P}} \left\{ \widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(g) \frac{f}{g} \right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p + \int_{\mathcal{P}} \left\{ \widetilde{\Gamma}_\mu f + \mathcal{K}_\mu(g) \frac{f}{g} \right\} \partial_p^\mu (\mathcal{D}^4 p). \tag{22}$$

Summing (21) and (22) and inserting the definition of $\mathcal{K}_\mu(g)$ Eq. (5) we obtain:

$$\mathcal{A}_2 = \int_{\mathcal{P}} \left\{ \widetilde{\Gamma}_\mu f + \left[I_\mu f - \partial_p^\nu (J_{\mu\nu} g) \frac{f}{g} \right] \right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p + \int_{\mathcal{P}} \left\{ \mathcal{K}_\mu(g) \frac{f}{g} - \mathcal{K}_\mu(f) \right\} \partial_p^\mu (\mathcal{D}^4 p). \tag{23}$$

Putting (19) and (23) together we get:

$$\mathcal{A}_1 + \mathcal{A}_2 = \int_{\mathcal{P}} \left\{ \partial_p^\nu (J_{\mu\nu} f) - \partial_p^\nu (J_{\mu\nu} g) \frac{f}{g} \right\} \partial_p^\mu \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p - \int_{\mathcal{P}} \widetilde{\Gamma}_\mu f \ln\left(\frac{f}{g}\right) \partial_p^\mu (\mathcal{D}^4 p) + \int_{\mathcal{P}} \left\{ \mathcal{K}_\mu(g) \frac{f}{g} - \mathcal{K}_\mu(f) \left[1 + \ln\left(\frac{f}{g}\right) \right] \right\} \partial_p^\mu (\mathcal{D}^4 p). \tag{24}$$

The third integral on the right-hand side of (24) contains two contributions and they both involve the contraction of the operator $\mathcal{K}$ with $\partial_p^\mu (\mathcal{D}^4 p)$. By Eq. (A9), this contraction is proportional to the contraction of $\mathcal{K}$ with p. By definitions (5)–(7), the action of this latter contraction on an arbitrary function h reads:

$$p^\mu \mathcal{K}_\mu(h) = p^\mu \{ I_\mu h - \partial_p^\nu (J_{\mu\nu} h) \} = D K^{\alpha\ \ \beta}_{\ \ \mu\ \ \nu} p^\mu \frac{p_\alpha p_\beta}{p \cdot U} (\partial_p^\nu h) + mc p^\mu F_\mu h. \tag{25}$$

The tensor $K^{\alpha\mu\beta\nu}$ is antisymmetric upon exchange of the indices μ and α, entailing that $K^{\alpha\mu\beta\nu} p_\alpha p_\mu p_\beta = 0$; moreover, the deterministic four-force F is orthogonal to the momentum p, i.e., $p^\mu F_\mu = 0$. Equation (25) therefore simply reduces to

$$p^\mu \mathcal{K}_\mu(h) = 0. \tag{26}$$

The last integral in (24) therefore disappears, and we can write:

$$\mathcal{A}_1 + \mathcal{A}_2 = \int_{\mathcal{P}} f \underbrace{\left\{ \frac{1}{f} \partial_p^\nu (J_{\mu\nu} f) - \frac{1}{g} \partial_p^\nu (J_{\mu\nu} g) \right\}}_{= J_{\mu\nu} D^\mu [f/g]} D^\mu [f/g] \, \mathcal{D}^4 p - \Gamma^\nu_{\mu\kappa} \int_{\mathcal{P}} p^\kappa p_\nu f \ln\left(\frac{f}{g}\right) \partial_p^\mu (\mathcal{D}^4 p), \tag{27}$$

where we used definition (14) of $D^\mu[\cdot]$ and definition (4) of $\widetilde{\Gamma}_\mu$.

Let us now address the $\mathcal{A}_3$ contribution to Eq. (18). Inserting the expression (A10) for $\partial_\kappa (\mathcal{D}^4 p)$, we have

$$\mathcal{A}_3 = -\int_{\mathcal{P}} p^\kappa f \ln\left(\frac{f}{g}\right) \partial_\kappa (\mathcal{D}^4 p) = \Gamma^\nu_{\kappa\mu} \int_{\mathcal{P}} p^\kappa p_\nu f \ln\left(\frac{f}{g}\right) \partial_p^\mu (\mathcal{D}^4 p) + \Gamma^\alpha_{\alpha\kappa} \int_{\mathcal{P}} p^\kappa f \ln\left(\frac{f}{g}\right) \mathcal{D}^4 p. \tag{28}$$

Inserting Eqs. (28) and (27) in (18), we obtain the wanted simple expression:

$$\nabla_\mu S^\mu_{f|g} = \int_{\mathcal{P}} J_{\mu\nu} D^\mu[f/g] D^\nu[f/g] \mathcal{D}^4 p. \tag{29}$$

□

2. The four-divergence of the entropy current is non-negative

We now state a second theorem, which, together with the previous one, will prove the *H*-theorem.

Theorem 2: *For any two arbitrary distributions f and g, the integrand in* Eq. (13) *of Theorem* 1 *is non-negative, that is*:

$$J_{\mu\nu} D^\mu[f/g] D^\nu[f/g] \geq 0. \tag{30}$$

Proof: Let us fix an arbitrary point x in space-time and choose as local reference frame $(\mathcal{R})$ at x the proper rest frame at x of the fluid surrounding the diffusing particle. By definition, in this reference frame, the components of the four-velocity $U(x)$ of the fluid at x are simply $U^\mu = (1/\sqrt{g_{00}})(1,0,0,0)$. Inserting these components into the definition (7) for J, we get

$$J^{00} = -\frac{D}{\sqrt{g_{00}} p_0} g^{ij} p_i p_j, \tag{31}$$

$$J^{0i} = -D\left(\frac{1}{g_{00}}(p_0)^2 g^{0i} - \frac{1}{g_{00}} p_0 g^{i\alpha} p_\alpha\right)\frac{\sqrt{g_{00}}}{p_0} = \frac{D}{\sqrt{g_{00}} p_0} g^{ij} p_0 p_j, \tag{32}$$

$$J^{ij} = -D\left(\frac{1}{g_{00}}(p_0)^2 g^{ij}\right)\frac{\sqrt{g_{00}}}{p_0} = -\frac{D}{\sqrt{g_{00}} p_0} g^{ij}(p_0)^2. \tag{33}$$

We thus find:

$$\begin{aligned} J^{\mu\nu} D_\mu D_\nu &= J^{00} D_0 D_0 + 2J^{0i} D_0 D_i + J^{ij} D_i D_j \\ &= -\frac{D}{\sqrt{g_{00}} p_0}\left[g^{ij} p_i p_j (D_0)^2 - 2g^{ij} p_0 p_j D_0 D_i + g^{ij}(p_0)^2 D_i D_j\right] \\ &= -\frac{D}{\sqrt{g_{00}} p_0} \underbrace{\left[p_i D_0 - (p_0)^2 D_i\right]}_{=v_i} g^{ij} \underbrace{\left[p_j D_0 - (p_0)^2 D_j\right]}_{=v_j} \\ &= -\frac{D}{\sqrt{g_{00}} p_0} g^{ij} v_i v_j. \end{aligned} \tag{34}$$

By Lemma 1 presented in the Appendix, the right-hand side of this equation is non-negative, which proves Theorem 2. □

IV. DISCUSSION

This article has been focused on the general relativistic Ornstein-Uhlenbeck process introduced in Ref. 6; we have constructed a conditional entropy four-current associated to any two arbitrary distributions solutions of Kolmogorov equation for the ROUP, and we have proven that the four-divergence of this current is always non-negative; this constitutes an *H*-theorem for the ROUP in curved space-time. It is a twofold generalization of the theorem introduced in Ref. 1. First, the *H*-theorem proved in Ref. 1 concerns flat space-time only. Second, Ref. 1 does not deal with a conditional entropy four-current associated to *two* arbitrary distributions, but only with the conditional entropy four-current of *one* arbitrary distribution with respect to the equilibrium distribution (invariant measure) of the ROUP in flat space-time. Let us note in this context that the ROUP does not generally admit an equilibrium distribution in curved space-time.[6]

We would like now to comment on this new H-theorem. Let us first remark that the theorem is valid in any Lorentzian space-time and for any time-like field U representing the velocity of the fluid in which the particles diffuse. In particular, the theorem is even valid in space-times with closed time-like curves, as the Gödel universe or the extended Kerr black hole,[12] and even if U is tangent to one of these closed time-like curves. The irreversibility measured by the local increase of the conditional entropy currents is entirely due to the Markovian character[25,11,23] of the ROUP and the remarkably general validity of the H-theorem proves that this irreversibility is in some sense stronger than all possible general relativistic chronological violations.

It should nevertheless be remarked that, as the Boltzmann-Gibbs entropy current associated to the relativistic Boltzmann equation, the conditional entropy four-currents introduced in Sec. III A are not necessarily time-like. And, even when they are time-like, their time-orientation in an orientable space-time generally depends on the point at which they are evaluated. Let us elaborate on this by first recalling the definition of the Boltzmann-Gibbs entropy current $S_{BG}[f]$ associated to a distribution f (see Ref. 14):

$$S_{BG}[f](x) = -\int_{\mathcal{P}} pf \ln f \mathcal{D}^4 p. \tag{35}$$

The normalization of f reads:

$$1 = \int_{\mathcal{T}_\Sigma} f d^3x \mathcal{D}^4 p, \tag{36}$$

where Σ is an arbitrary space-like hypersurface of the space-time $\mathcal{M}$ and where $\mathcal{T}_\Sigma \subset T^*(\mathcal{M})$ is defined by

$$\mathcal{T}_\Sigma = \{(x,p) \in T^*(\mathcal{M}), x \in \Sigma\}. \tag{37}$$

As a probability distribution, f is certainly non-negative; but f may take values both superior and inferior to unity. Therefore, nothing can be said on the sign of the function $f \ln f$ against which the time-like vector p is integrated in (35). This entails that $S_{BG}[f](x)$ may be either time-like or space-like. Also note that the sign of the zeroth component of $S_{BG}[f](x)$ cannot be ascertained either; thus, even when time-like, the Boltzmann-Gibbs entropy current may be past as well as future oriented (in a time-orientable space-time).

Similarly, the sign of the function $f(x,p)\ln(f(x,p)/g(x,p))$ appearing in definition (12) of the conditional entropy current $S_{f|g}(x)$ generally depends on p (and x) and $S_{f|g}(x)$ may therefore not be time-like. For the same reason, the sign of the zeroth component of $S_{f|g}(x)$ also generally depends on the point in space-time so that the conditional entropy currents, even when time-like, may not have a definite time-orientation (in a time-orientable space-time).

The Galilean limit deserves a particular discussion. The very notions of time-like and space-like vector-fields do not exist in this limit and only the time-orientation of the conditional entropy currents should be addressed. In the Galilean limit, the zeroth component of $S_{f|g}(x)$ reads

$$s_{f|g}(t,\mathbf{x}) = -\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{p}) \ln\left(\frac{f(t,\mathbf{x},\mathbf{p})}{g(t,\mathbf{x},\mathbf{p})}\right) d^3p; \tag{38}$$

note that this expression coincides with the conditional entropy density of the usual, non relativistic Ornstein-Uhlenbeck process.[22] A reasoning similar to the one presented in the preceding paragraph shows that this density may take positive as well as negative values. The time-orientation of the conditional entropy currents is therefore generally position-dependent, even in the Galilean regime.

However, in the Galilean limit, it surely makes sense to integrate $s_{f|g}(t,\mathbf{x})$ over the whole three-dimensional (3D) space to obtain the total (time-dependent) conditional entropy $\mathcal{S}(t)$ of f with respect to g and this quantity can be proven to be non-positive. The proof[22,3] is based on the so-called Gibbs-Klein inequality[25]

$$F \ln F \geqslant F - 1, \tag{39}$$

valid for any positive real number F and applied to $F(t,\mathbf{x},\mathbf{p}) = f(t,\mathbf{x},\mathbf{p})/g(t,\mathbf{x},\mathbf{p})$ (with the hypothesis that g does not vanish anywhere in $\mathbb{R}^3$). One has indeed:

$$\int_{\mathcal{V}} s_{f|g}(t,\mathbf{x}) d^3x = -\int_{\mathcal{V}\times\mathbb{R}^3} f(t,\mathbf{x},\mathbf{p}) \ln\left(\frac{f(t,\mathbf{x},\mathbf{p})}{g(t,\mathbf{x},\mathbf{p})}\right) d^3x\, d^3p \leqslant \int_{\mathcal{V}\times\mathbb{R}^3} \big(f(t,\mathbf{x},\mathbf{p}) - g(t,\mathbf{x},\mathbf{p})\big) d^3x\, d^3p \leqslant 0. \tag{40}$$

This calculation can be extended formally to the special and general relativistic situations, but, since conditional entropy four-currents are then not necessarily time-like, their integrals on space-like 3D submanifolds may take positive or negative values. It is therefore far from clear that the concept of *total* conditional entropy makes sense in the relativistic regime. In particular, the relativistic H-theorem proved in this article should be primarily considered as a purely local result.

Thus, the conceptual status of the entropy currents introduced in Sec. III A is in a certain sense similar to the status of the general relativistic black hole entropies.[28,29,17,15] Indeed, we have shown in this article that stochastic processes theory proves the existence of conditional entropy currents in curved space-time and permits their computation, exactly as quantum field theory and string theory both prove the existence of black-holes entropies and furnish the tools necessary for their computations. But the standard statistical interpretation of conditional entropy currents via their fluxes through 3D space-like submanifolds is certainly not straightforward in curved space-time, as the usual interpretation of entropy and temperature via Gibbs canonical ensembles does not seem to extend smoothly to black hole thermodynamics.[29]

It is our opinion that progress in interpreting the notion of entropy in curved space-time can best be achieved by studying specific examples in particular circumstances where most results can be obtained by explicit or semi-explicit calculations. The ROUP is obviously an interesting tool for such computations and diffusion in space-times exhibiting naked or unnaked singularities should certainly be studied in detail.

Finally, it would naturally be most interesting to determine if H-theorems can also be proved for the two "new" relativistic stochastic processes recently proposed as alternative models of relativistic diffusion in Refs. 10 and 9.

APPENDIX

General relations

A basic assumption of general relativity is that the connection ∇ used in space-time is the Levi-Civita connection of the space-time metric g.[27] Given a coordinate basis, this translates into the following relation between the metric components $g_{\mu\nu}$ and the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$:

$$\partial_\kappa g_{\mu\nu} = \Gamma^{\alpha}_{\kappa\mu} g_{\alpha\nu} + \Gamma^{\alpha}_{\kappa\nu} g_{\mu\alpha}. \tag{A1}$$

Another equivalent form of (A1) is

$$\partial_\kappa g^{\mu\nu} = -\Gamma^{\mu}_{\kappa\alpha} g^{\alpha\nu} - \Gamma^{\nu}_{\kappa\alpha} g^{\mu\alpha}. \tag{A2}$$

A direct consequence of (A2) is that, for any vector p:

$$(\partial_\kappa g^{\mu\nu}) p_\mu p_\nu = -\Gamma^{\mu}_{\kappa\alpha} p^\alpha p_\mu - \Gamma^{\nu}_{\kappa\alpha} p^\alpha p_\nu = -2\Gamma^{\nu}_{\kappa\mu} p_\nu p^\mu. \tag{A3}$$

Another useful relation reads:[20]

$$\partial_\kappa \det g = (\det g) g^{\mu\nu} \partial_\kappa g_{\mu\nu}. \tag{A4}$$

Using (A1), this translates into

$$\partial_\kappa \det g = (\det g) g^{\mu\nu} 2\Gamma^\alpha_{\kappa\mu} g_{\alpha\nu} = 2(\det g)\Gamma^\alpha_{\kappa\alpha}. \tag{A5}$$

A useful lemma

Lemma 1: *Let (∂_μ) be a (local) coordinate basis of a Lorentzian space-time (with time-like ∂_0). Then, at any point x of space-time, the set of the six spatial components $g^{ij}(x)$ of the inverse metric tensor define a non-positive quadratic form. More precisely,*

$$g^{ij}(x) v_i v_j \leqslant 0 \quad \text{for all } (v_1, v_2, v_3) \in \mathbb{R}^3.\text{[1]} \tag{A6}$$

Proof: Let x be a point in space-time and suppose there exists a set of three real numbers (v_1, v_2, v_3) such that $g^{ij}(x) v_i v_j > 0$. Define V, cotangent to the space-time manifold at x, by its components $V_0 = 0$, $V_1 = v_1$, $V_2 = v_2$, $V_3 = v_3$. The vector V is both time-like and orthogonal to ∂_0. The space cotangent to the space-time manifold at x therefore admits a time-like subspace of dimension at least two, which is impossible for a Lorentzian space-time. This proves the lemma.□

Derivatives of the volume measure in momentum-space

Let us now evaluate the partial derivatives of the volume measure $\mathcal{D}^4 p$ with respect to both space-time coordinates and momentum components. The measure $\mathcal{D}^4 p$ is defined by an expression which involves the product of a Heaviside function and a Dirac distribution. Direct derivation of this expression would lead to a product of Dirac distributions, which is not a well-defined mathematical object. To avoid this (at least formal) problem, we introduce a class of regular functions h_ϵ, which uniformly converge towards δ as ϵ tends to zero and write:

$$\begin{aligned}\partial_p^\mu\{\theta(p_0)\,\delta(p^2 - m^2c^2)\} &= \lim_{\epsilon\to 0} \partial_p^\mu\{\theta(p_0) h_\epsilon(g^{\alpha\beta} p_\alpha p_\beta - m^2c^2)\} \\ &= \lim_{\epsilon\to 0}\{\delta(p_0)\,\delta_0^\mu h_\epsilon(g^{\alpha\beta} p_\alpha p_\beta - m^2c^2) + \theta(p_0)\partial_p^\mu[h_\epsilon(g^{\alpha\beta} p_\alpha p_\beta - m^2c^2)]\} \\ &= \lim_{\epsilon\to 0}\{\delta(p_0)\,\delta_0^\mu h_\epsilon(g^{ij} p_i p_j - m^2c^2) + \theta(p_0) 2 g^{\mu\nu} p_\nu h'_\epsilon(g^{\alpha\beta} p_\alpha p_\beta - m^2c^2)\}.\end{aligned} \tag{A7}$$

By Lemma 1 (Eq. (A6)), $g^{ij} p_i p_j \leqslant 0$. The argument of h_ϵ in the last line of (A7) is therefore always strictly negative. The term involving h_ϵ thus disappears for $\epsilon \to 0$ and we are left with the result:

$$\partial_p^\mu\{\theta(p_0)\,\delta(p^2 - m^2c^2)\} = 2p^\mu\,\theta(p_0)\,\delta'(p^2 - m^2c^2). \tag{A8}$$

This equation leads directly to the following expression for the partial derivatives of $\mathcal{D}^4 p$ with respect to momentum components:

$$\partial_p^\mu(\mathcal{D}^4 p) = \partial_p^\mu\left\{\theta(p_0)\,\delta(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}}\right\} d^4p = 2p^\mu\,\theta(p_0)\,\delta'(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}} d^4p. \tag{A9}$$

[1] See for example Sec. 84 of Ref. 20.

Let us now focus on the derivatives of $\mathcal{D}^4 p$ with respect to space-time coordinates. Using Eqs. (A3), (A5), and (A9), we obtain

$$\partial_\kappa(\mathcal{D}^4 p) = \partial_\kappa\left\{ \theta(p_0)\,\delta(g^{\mu\nu}p_\mu p_\nu - m^2c^2)\frac{1}{\sqrt{-\det g}}\right\} d^4p = \theta(p_0)(\partial_\kappa g^{\mu\nu})p_\mu p_\nu \delta'(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}}d^4p$$
$$+ \theta(p_0)\,\delta(p^2 - m^2c^2)\partial_\kappa\left(\frac{1}{\sqrt{-\det g}}\right)d^4p = -2\Gamma^\nu_{\kappa\mu}p_\nu p^\mu \theta(p_0)\,\delta'(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}}d^4p$$
$$- \theta(p_0)\,\delta(p^2 - m^2c^2)\frac{1}{\sqrt{-\det g}}\frac{\partial_\kappa \det g}{2\det g}d^4p = -\Gamma^\nu_{\kappa\mu}p_\nu\partial^\mu_p(\mathcal{D}^4p) - \Gamma^\alpha_{\kappa\alpha}\mathcal{D}^4p. \tag{A10}$$

[1] Barbachoux, C., Debbasch, F., and Rivet, J. P., "Covariant Kolmogorov equation and entropy current for the relativistic Ornstein-Uhlenbeck process," Eur. Phys. J. B **23**, 487 (2001).
[2] Barbachoux, C., Debbasch, F., and Rivet, J. P., "The spatially one-dimensional relativistic Ornstein-Uhlenbeck process in an arbitrary inertial frame," Eur. Phys. J. B **19**, 37 (2001).
[3] Beck, C. and Schloegl, F., *Thermodynamics of Chaotic Systems, An Introduction* (Cambridge University Press, Cambridge, 1993).
[4] Boltzmann, L., *Vorlesungen über Gastheorie*, Erweiterter Nachruck der 1896-1898 bei Ambrosius Barth in Leipzig erschienen Ausgabe, 1981. Akademische Druck u. Verlagsanstalt, Graz.
[5] de Groot, S. R., van Leeuwen, W. A., and van Weert, C. G., *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
[6] Debbasch, F., "A diffusion process in curved space-time," J. Math. Phys. **45**, 7 (2004).
[7] Debbasch, F., Mallick, K., and Rivet, J. P., "Relativistic Ornstein-Uhlenbeck process," J. Stat. Phys. **88**, 945 (1997).
[8] Debbasch, F. and Rivet, J. P., "A diffusion equation from the relativistic Ornstein-Uhlenbeck process," J. Stat. Phys. **90**, 1179 (1998).
[9] Dunkel, J. and Hänggi, P., "Theory of relativistic Brownian motion: The (1+1)-dimensional case," Phys. Rev. E **71**, 016124 (2005).
[10] Franchi, J. and le Jan, Y., "Relativistic diffusions and Schwarzschild geometry," arXiv math.PR/0410485, 2004.
[11] Grimmett, G. R. and Stirzaker, D. R., *Probability and Random Processes*, 2nd ed. (Oxford University Press, Oxford, 1994).
[12] Hawking, S. W. and Ellis, G. F. R., *The Large Scale Structure of Space-Time*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1973).
[13] Huang, K., *Statistical Machanics*, 2nd ed. (Wiley, New York, 1987).
[14] Israel, W., "*Covariant fluid mechanics and thermodynamics: An introduction*," in Relativistic Fluid Dynamics, edited by A. Anile, and Y. Choquet-Bruhat, Lecture Notes in Mathematics, Vol. 1385 (Springer, Berlin, 1987).
[15] Johnson, C. V., *D-Branes* (Cambridge University Press, Cambridge, 2003).
[16] Jüttner, F., "Die relativistische quantentheorie des idealen gases," Z. Phys. **47**, 542–566 (1928).
[17] Kaku, M., *Introduction to Superstrings and M-theory*, 2nd ed. (Springer, New York, 1999).
[18] Kullback, S., *Information Theory and Statistics* (Wiley, New York, 1951).
[19] Kullback, S. and Leibler, R. A., "On information and sufficiency," Ann. Math. Stat. **22**, 79–86 (1951).
[20] Landau, L. D. and Lifshitz, E. M., *The Classical Theory of Fields*, 4th ed. (Pergamon, Oxford, 1975).
[21] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics* (Pergamon, Oxford, 1987).
[22] Mackey, M. C., *Time's Arrow: The Origins of Thermodynamic Behavior* (Springer, Berlin, 1992).
[23] Øksendal, B., *Stochastic Differential Equations*, 5th ed. (Universitext. Springer, Berlin, 1998).
[24] Truesdell, C. A., *Rational Thermodynamics* (Springer, New York, 1984).
[25] van Kampen, N. G., *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[26] Voigt, J., "Stochastic operators, information, and entropy," Commun. Math. Phys. **81**, 31–38 (1981).
[27] Wald, R. M., *General Relativity* (The University of Chicago Press, Chicago, 1984).
[28] Wald, R. M., *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics, Chicago Lectures in Physics* (The University of Chicago Press, Chicago, 1994).
[29] Wald, R. M., editor, *Black Holes and Relativistic Stars* (The University of Chicago Press, Chicago, 1998).